Existence and Comparisons for BSDEs in general spaces

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Outline

Classical BSDEs

A BSDE is an equation of the form

$$Y_t - \int_{]t,T]} F(\omega, u, Y_u, Z_u)du + \int_{]t,T]} Z_u dW_u = Q$$

where the solution pair (Y, Z) is adapted, Z is predictable and Q is some $\mathcal{F}_T$-measurable random variable.

- These equations have been studied in depth over the last 20 years.
- They have significant applications in Optimal Control and Mathematical Finance.
- My interest is on generalising these equations to allow for different types of filtrations and randomness.

BSDEs in Discrete time

Recent work has considered BSDEs in discrete time, finite state systems

$$Y_t - \sum_{t \leq u < T} F(\omega, u, Y_u, Z_u) + \sum_{t \leq u < T} Z_u dM_u = Q.$$

where M is a $\mathbb{R}^N$-valued martingale defining the filtration

- Existence and comparison results can be obtained for these equations
- These equations form a complete representation of various time-consistent operators on $L^0(\mathcal{F}_T)$.
- *Is there a way to unite this discrete time theory with the classical one?*

BSDEs in general spaces

- Today we will consider BSDEs where both the martingale and driver terms can jump.
- This will include, as special cases, both the discrete time and continuous time theory of BSDEs
- Very few assumptions are needed on the underlying probability space.

Our first step is to state a general form of the Martingale representation theorem...

Theorem (Davis & Varaiya 1974)

Let $(\Omega, \mathcal{F}_T, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$ *be a filtered probability space. Suppose* $L^2(\mathcal{F}_T)$ *is separable. Then there exists a sequence of martingales* $M^1, M^2 \ldots$ *such that any martingale N can be written as*

$$N_t = N_0 + \sum_{i=1}^{\infty} \int_{]0,t]} Z_u^i dM_u^i$$

for some predictable processes Z^i*, and*

$$\langle M^1 \rangle \succ \langle M^2 \rangle \succ \ldots$$

as measures on $\Omega \times [0, T]$*.*

i.e. $\langle M^i \rangle(A) = E[\int_{[0,T]} I_A d\langle M^i \rangle]$ for $A \subseteq \Omega \times [0, T]$.

Definition

For μ a fixed nonnegative Stieltjes measure with $\mathbb{P} \times \mu \succ \langle M^1 \rangle$, let $\|\cdot\|_{M_t}$ be the stochastic seminorm on infinite $\mathbb{R}^K$-valued sequences given by

$$\|(z^1, z^2, \dots)\|_{M_t}^2 = \sum_{i=0}^{\infty} \|z^i\|^2 \frac{d\langle M^i \rangle_t}{d(\mathbb{P} \times \mu_t)}.$$

We shall assume that some deterministic Stieltjes measure μ with $\mathbb{P} \times \mu \succ \langle M^1 \rangle$ exists.

BSDEs in general spaces

Consider an equation of the form:

$$Y_t - \int_{]t,T]} F(\omega, u, Y_{u-}, \mathbf{Z}_u) d\mu + \sum_{i=1}^{\infty} \int_{]t,T]} Z_u^i dM_u^i = Q$$

where

- $Q \in L^2(\mathcal{F}_T)$,
- $Y \in \mathbb{R}^K$ is adapted and $\sup_{t\in[0,T]}\{\|Y_t\|^2\} < \infty$,
- $\mathbf{Z}_t \equiv (Z^1, Z^2, \ldots)$ is a sequence of predictable $\mathbb{R}^K$-valued processes such that $\mathbf{Z} \in \mathcal{H}_M^2$, that is

$$E\left[\sum_i \int_{]0,T]} \|Z_t^i\|^2 d\langle M^i\rangle_t\right] = \int_{]0,T]} E\left[\|\mathbf{Z}_u\|_{M_u}^2\right] d\mu_t < \infty$$

BSDEs in general spaces

$$Y_t - \int_{]t,T]} F(\omega, u, Y_{u-}, \mathbf{Z}_u) d\mu + \sum_{i=1}^{\infty} \int_{]t,T]} Z_u^i dM_u^i = Q$$

Also,

- μ is a deterministic Stieltjes measure on $[0, T]$. For simplicity, assume μ is nonnegative and $\mathbb{P} \times \mu \succ \langle M^1 \rangle$.
- F is a progressively measurable function such that $F(\omega, t, 0, \mathbf{0})$ is μ-square-integrable.

Existence result

Theorem

Suppose F is firmly Lipschitz, that is, there exists a constant c and a map $c_{(\cdot)} : [0, T] \to [0, c]$ such that

$$\|F(\omega, t, y, \mathbf{z}) - F(\omega, t, y', \mathbf{z}')\|^2 \leq c_t \|y - y'\|^2 + c\|\mathbf{z} - \mathbf{z}'\|^2_{M_t}$$

and

$$c_t(\Delta\mu_t)^2 < 1.$$

Then the BSDE has a unique solution, (up to indistinguishability if $d\mu \succ dt$).

- As the discrete time BSDE can be embedded in continuous time, and the necessary and sufficient condition for existence in discrete time is that $y \mapsto y - F(\omega, t, y, z)$ is a bijection, the classical requirement of Lipschitz continuity is clearly insufficient.
- On the other hand, if μ is continuous, then these assumptions are simply classical Lipschitz continuity.
- By the use of the Radon-Nikodym theorem for measures on $\Omega \times [0, T]$, the requirement that μ is deterministic, nonnegative and $\mathbb{P} \times \mu \succ \langle M^1 \rangle_t$ is a flexible one, as exceptions can be instead incorporated into F.

- From a mathematical perspective, this unites the theory of BSDEs in discrete and continuous time.
- From a modelling perspective, it allows us to build models without quasi-left-continuity.
  - For interest rate modelling, when central bank decisions are announced on certain dates.
  - For evaluating contracts where some counterparty decisions must be made on a certain date.
- Allowing these discontinuities is one step closer to a general semimartingale theory of BSDEs.

We now proceed to the proof of existence and uniqueness.

Definition (Stieltjes-Doleans-Dade Exponentials)

For any cadlag function of finite variation ν, let

$$\mathfrak{E}(\nu; t) = e^{\nu_t} \prod_{0 \le s \le t} (1 + \Delta\nu_s) e^{-\Delta\nu_s}.$$

and if $\Delta\nu_s < 1$ a.s.

$$\tilde{\nu}_t = \nu_t + \sum_{0 \le s \le t} \frac{(\Delta\nu_s)^2}{1 - \Delta\nu_s} \quad \text{and} \quad \mathfrak{E}(-\nu; t) = \mathfrak{E}(\tilde{\nu}; t)^{-1}.$$

Lemma (Backwards Grönwall inequality with jumps)

For semimartingales u, w, a finite-variation process ν with $\Delta\nu_s < 1$ a.s., if

$$du_t \ge -u_t d\nu_t + dw_t$$

then

$$d(u_t \mathfrak{E}(\tilde{\nu}; t)) \ge (1 - \Delta\nu_t)^{-1} \mathfrak{E}(\tilde{\nu}; t-) dw_t.$$

Lemma (Bound on BSDE solutions)

Let Y be a solution to a BSDE with firm Lipschitz driver, and let $Z \in \mathcal{H}^2_M$. Then $E[\sup_{t\in[0,T]}\{\|Y_t\|^2\}] < \infty$ if and only if

$$\int_{]0,T]} E[\|Y_{t-}\|^2]d\mu < \infty.$$

Lemma (BSDEs, no dependence on $Y, \mathbf{Z}$)

Let $F : \Omega \times [0, T] \to \mathbb{R}^K$. Then a BSDE with driver F has a solution.

Proof.

Simple application of martingale representation theorem. □

Sketch proof of existence theorem

Assume $\mu_T \leq 1$ and $c_t \Delta\mu_t < 1$. We have the following bound:

Lemma

For two BSDEs with solutions Y, Y', etc. let $\delta Y := Y - Y'$,
$\delta \mathbf{Z} := \mathbf{Z} - \mathbf{Z}'$, $\quad \delta_2 f_t = F(\omega, t, Y'_{t-}, \mathbf{Z}'_t) - F'(\omega, t, Y'_{t-}, \mathbf{Z}'_t)$.
For measurable functions $x, w : [0, T] \to [0, \infty]$ with $\Delta\mu_t \leq x_t^{-1}$,

$$dE[\|\delta Y_t\|^2] \geq -E[\|\delta Y_t\|^2] d\upsilon_t + E[\|\delta \mathbf{Z}_t\|^2_{M_t}](1 - \Delta\upsilon_t) d\rho_t \\ - E[\|\delta_2 f_t\|^2](1 - \Delta\upsilon_t) d\pi_t.$$

where

$$d\upsilon_t = [(x_t^{-1} - \Delta\mu_t)(1 + w_t)c_t + x_t] d\mu_t$$
$$d\pi_t = [(x_t^{-1} - \Delta\mu_t)(1 + w_t^{-1})](1 - \Delta\upsilon_t)^{-1} d\mu_t$$
$$d\rho_t = [1 - (x_t^{-1} - \Delta\mu_t)(1 + w_t)c](1 - \Delta\upsilon_t)^{-1} d\mu_t$$

Combining this bound with our backwards Grönwall inequality gives, provided $\Delta \upsilon_t < 1$,

$$\begin{aligned} &E[\|\delta Y_t\|^2]\mathfrak{E}(\tilde{\upsilon}; t) + \int_{]t,T]} E[\|\delta Z_s\|^2]\mathfrak{E}(\tilde{\upsilon}; s-)d\rho_s \\ &\qquad \leq E[\|\delta Q\|^2]\mathfrak{E}(\tilde{\upsilon}; T) + \int_{]t,T]} E[\|\delta_2 f_s\|^2]\mathfrak{E}(\tilde{\upsilon}; s-)d\pi_s. \end{aligned} \tag{1}$$

Take a left limit in t, then evaluate the $d\mu$ integral on $]0, T]$,

$$\begin{aligned} &\int_{]0,T]} E[\|\delta Y_{t-}\|^2]\mathfrak{E}(\tilde{\upsilon}; t-)d\mu_t + \int_{]0,T]} \mu_s E[\|\delta Z_s\|^2]\mathfrak{E}(\tilde{\upsilon}; s-)d\rho_s \\ &\qquad \leq \mu_T E[\|\delta Q\|^2]\mathfrak{E}(\tilde{\upsilon}; T) + \int_{]0,T]} \mu_s E[\|\delta_2 f_s\|^2]\mathfrak{E}(\tilde{\upsilon}; s-)d\pi_s \end{aligned} \tag{2}$$

We now construct Picard iterates to solve the BSDE with driver $F(\cdot,\cdot,Y^0,\cdot)$. For any approximation $(Y^n,\mathbf{Z}^n)$, let $(Y^{n+1},\mathbf{Z}^{n+1})$ be the solution to the BSDE with driver $F(\cdot,\cdot,Y^0,\mathbf{Z}^n)$ and terminal value Q.

If $\delta Y^n,\delta\mathbf{Z}^n$ denote the difference between two approximations, taking $w_t=1, x_t^{-1}=\frac{1}{4c}+\Delta\mu_t$, inequality (1) reduces to

$$\begin{aligned}\int_{]0,T]} & E[\|\delta Z_s^{n+1}\|^2]\mathfrak{E}(\tilde{\upsilon};s-)(1-\Delta\upsilon_s)^{-1}d\mu_s \\ &\leq \frac{1}{2}\int_{]0,T]} E[\|\delta Z_s^{n}\|^2]\mathfrak{E}(\tilde{\upsilon};s-)(1-\Delta\upsilon_s)^{-1}d\mu_s\end{aligned}$$

and so the contraction mapping theorem applies, under an equivalent norm.

Using this, similarly construct iterates $(Y^n, \mathbf{Z}^n)$ each solving the BSDE with driver $F(\cdot, \cdot, Y^n, \cdot)$, terminal value Q.
If $c_t \Delta\mu_t < 1$ then $c_t \Delta\mu_t < 1 - \epsilon$ for some $\epsilon > 0$. Let

$$x_t = \frac{c(1 + 2\epsilon^{-1})}{1 + (1 + 2\epsilon^{-1})\Delta\mu_t}; \qquad w_t = 3\epsilon^{-1}.$$

From inequality (2),

$$\int_{]0,T]} E[\|\delta Y^{n+1}_{t-}\|^2]\mathfrak{E}(\tilde{v}; t-) d\mu_t \\ \leq \left(1 - \frac{\epsilon^2}{4}\right) \int_{]0,T]} E[\|\delta Y^{n}_{t-}\|^2]\mathfrak{E}(\tilde{v}; t-) d\mu_t$$

again the contraction mapping theorem gives the existence of a unique solution.

Now relax the restrictions $\mu_T \leq 1$ and $c_t \Delta\mu_t < 1$ to the assumption

$$c_t(\Delta\mu_t)^2 < 1.$$

If

$$d\nu_t = \frac{2(1+\epsilon^{-1})c}{\epsilon + 2(1+\epsilon^{-1})c\Delta\mu_t} d\mu_t = \lambda_t^{-1} d\mu_t$$

we have $\Delta\nu_t < 1$ and $c_t\lambda_t^2\Delta\nu_t < 1$. Then for some $\eta > 0$ there is a finite partition $\{0 = t_0, t_1, ..., t_B = T\}$ with $\nu(]t_i, t_{t+1}]) \leq 1 - \eta$ for all i.
Write

$$\nu_t^k = \int_{]0, t\wedge t_{k+1}]} \left[\frac{\eta}{\nu_{t_k}} + \left(1 - \frac{\eta}{\nu_{t_k}}\right) I_{t>t_k}\right] d\nu_t$$

Using the Radon-Nikodym theorem, write the BSDE in terms of ν_T^k, then we have a unique solution, which agrees with our original BSDE on $[t_{B-1}, t_B]$.
Use backwards induction to construct the solution for all times.

Comparison results

With our existence theory, we now wish to be able to compare solutions to BSDEs.

- As our martingales can jump, we need to be careful.
- A comparison result is closely related to a nonlinear no-arbitrage result, so similar language may be helpful.

For simplicity, we shall consider the scalar case only.

Balanced drivers

Definition

Let F be such that for any square-integrable Y, any $\mathbf{Z}, \mathbf{Z}' \in \mathcal{H}^2_M$,

$$-\int_{]0,t]} [F(\omega, u, Y_{u-}, \mathbf{Z}_u) - F(\omega, u, Y_{u-}, \mathbf{Z}'_u)] d\mu_u + \sum_i \int_{]0,t]} [(Z)^i_u - (Z')^i_u] dM^i_u$$

has an equivalent martingale measure. Then F shall be called *balanced*.

Comparison Theorem

Theorem

Let $(Y, \mathbf{Z})$ and $(Y', \mathbf{Z}')$ be the solutions to two BSDEs with drivers F, F' and terminal conditions Q, Q'. Then if

- $Q \geq Q'$ *a.s.*
- $F(\omega, t, Y'_{t-}, Z'_t) \geq F'(\omega, t, Y'_{t-}, Z'_t)$ $\mu \times \mathbb{P}$*-a.s. and*
- *F is balanced*

It follows that $Y_t \geq Y'_t$ for all t. The strict comparison also applies.

Sketch proof

Omit ω, t for clarity. Decompose $Y - Y'$ into the differences based on

- $Q - Q'$ (nonnegative),
- $F(Y', \mathbf{Z}') - F'(Y', \mathbf{Z}')$ (nonnegative),
- $F(Y', \mathbf{Z}) - F(Y', \mathbf{Z}')$ (equivalent martingale measure),
- $F(Y, \mathbf{Z}) - F(Y', \mathbf{Z})$ (remainder).

By assumption and the existence of a martingale measure $\tilde{\mathbb{P}}$, this implies

$$Y_t - Y'_t - E_{\tilde{\mathbb{P}}}\left[\int_{]t,T]} F(Y_{u-}, \mathbf{Z}_u) - F(Y'_{u-}, \mathbf{Z}_u) d\mu \middle| \mathcal{F}_t\right] \geq 0$$

Lipschitz continuity and another form of Backwards Grönwall inequality, applied on the set $Y_t - Y'_t \leq 0$, then gives the result.

- These conditions are the natural extension of the requirements in discrete time, which can be shown to be (loosely) necessary for the general result to hold.
- As the comparison theorem is the non-linear version of a no-Arbitrage result, it is natural to think of it in terms of equivalent-martingale-measures.
- This also indicates that, perhaps with generalisation to local- or σ-martingales, it may be the most general condition to use.
- The various classical examples of the comparison theorem can all be seen to be special cases of this requirement.

Nonlinear Expectations

We can now construct examples of nonlinear expectations in these general probability spaces.

Theorem

Let F be a firmly Lipschitz driver. Define $\mathcal{E}_t(Q) = Y_t$, where Y is the solution to the BSDE with driver F, terminal value Q. Then

- $\mathcal{E}_s(\mathcal{E}_t(Q)) = \mathcal{E}_s(Q)$ *for all* $t \geq s$.
- $I_A\mathcal{E}_t(I_A Q) = I_A\mathcal{E}_t(Q)$ *for all* $A \in \mathcal{F}_t$.
- *If F is balanced, then $Q \geq Q'$ a.s. implies $\mathcal{E}_t(Q) \geq \mathcal{E}_t(Q')$.*
- *If $F(\omega, t, y, \mathbf{0}) = 0$ then $\mathcal{E}_t(Q) = Q$ for all $Q \in L^2(\mathcal{F}_t)$.*
- *If F is independent of y, then $\mathcal{E}_t(Q + q) = \mathcal{E}_t(Q) + q$ for all $q \in L^2(\mathcal{F}_t)$.*
- *If F is balanced and concave, then $\mathcal{E}$ is concave.*

Conclusions

We have presented a theory of BSDEs in general probability spaces

- Our only assumptions are that $L^2(\mathcal{F}_T)$ is separable, and that a Stieltjes measure μ with $\mathbb{P} \times \mu \succeq \langle M^1 \rangle$ exists.
- This unites the discrete and continuous theories of BSDEs.
- We have conditions for existence of unique solutions of BSDEs in this context, based on Lipschitz continuity.
- We have a version of the comparison theorem for this situation.
- This allows modelling of various situations with less continuity than classically required.