Duality for set-valued measures of risk

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With: **F. Heyde (Halle), B. Rudloff & M. Yankova (Princeton)**

‖ ▶ **Basic question.**

How to evaluate the risk of $X \in L_d^0 = L^0\left(\Omega, \mathcal{F}, P; \mathbb{R}^d\right)$?

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(1) $u^1, u^2 \in \mathbb{R}^d$ compensate for the risk of X, but might not be comparable.

Example. 1-1 exchange rate, 10% transaction costs: neither of

$$u^1 = \begin{pmatrix} 1000 \\ 0 \end{pmatrix}, \quad u^2 = \begin{pmatrix} 0 \\ 1000 \end{pmatrix}$$

is "better".

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How to evaluate the risk of $X \in L_d^0 = L^0\left(\Omega, \mathcal{F}, P; \mathbb{R}^d\right)$?

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(1) $u^1, u^2 \in \mathbb{R}^d$ compensate for the risk of X, but might not be comparable.

(2) $u^1 \in \mathbb{R}^d$ does not compensate for the risk of X, but can be exchanged at initial time into $u^2 \in \mathbb{R}^d$ which does.

(3) $u \in \mathbb{R}^d$ does not compensate for the risk of X^1, but X^1 can be exchanged at terminal time into X^2 such that u compensates for X^2.

▶ Basic idea.

$A \subseteq L_d^0$ set of acceptable payoffs: The mapping

$$X \mapsto R_A(X) = \left\{u \in \mathbb{R}^d \colon X + u\mathbb{1} \in A\right\} \subseteq \mathcal{P}\left(\mathbb{R}^d\right)$$

is understood as a set-valued risk measure $R_A \colon L_d^0 \to \mathcal{P}\left(\mathbb{R}^d\right)$.

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is understood as a set-valued risk measure $R_A \colon L_d^0 \to \mathcal{P}\left(\mathbb{R}^d\right)$.

$\|\blacktriangleright$ **References.**

Superhedging theorems for markets with transaction costs (Kabanov 99, Schachermayer 04, Pennanen/Penner 10 ...)

Set-valued risk measure ad hoc: Jouini/Touzi/Meddeb 04

Complete theory, constant cone: Hamel/Heyde 10

Complete theory, random cone: Hamel/Heyde/Rudloff 10+

‖▶ **Rest of the talk.**

- Formal definitions and primal representation
- Dual representation and dual variables
- Super-hedging price as a coherent SRM
- A set-valued AV@R: definition and computation

|| ▶ Formal definitions.

Space of eligible portfolios.

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Acceptance sets. $A \subseteq L^p_d$, $0 \leq p \leq \infty$, with

(A1) $M\mathbb{1} \cap A \neq \emptyset$, $M\mathbb{1} \cap \left(L^p_d \backslash A\right) \neq \emptyset$

(A2) $A + \left(L^p_d\right)_+ \subseteq A$.

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Risk measures. $R_A \colon L^p_d \to \mathcal{P}\left(M\right)$ defined by

$$R_A\left(X\right) = \left\{u \in M \colon X + u\mathbb{1} \in A\right\}, \quad X \in L^p_d.$$

Note. Set-valuedness solves the problem of incomparableness!

Result. The set-valued function $X \mapsto R_A(X)$ is

(R0) M-translative, i.e.

$$\forall X \in L^p_d,\ \forall u \in M\colon R(X + u\mathbb{1}) = R(X) - u.$$

(R1) finite at zero: $R(0) \neq \emptyset$ and $R(0) \neq M$.

(R2) $\left(L^p_d\right)_+$-monotone, i.e.

$$X^2 - X^1 \in \left(L^p_d\right)_+ \quad \Rightarrow \quad R\left(X^2\right) \supseteq R\left(X^1\right).$$

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M-translative functions and some subsets of L_d^p are one–to–one via

$$A_R = \left\{X \in L_d^p\colon 0 \in R(X)\right\}, \quad R_A(X) = \{u \in M\colon X + u\mathbb{1} \in A\}$$

Conical market models with one period.

At Initial Time.

- $K_I \subseteq \mathbb{R}^d$ a solvency cone: closed convex cone with $\mathbb{R}^d_+ \subseteq K_I \neq \mathbb{R}^d$
- $K_I^M = K_I \cap M$ solvency cone restricted to eligible portfolios

K_I**-compatible:** $X \in A$, $u \in K_I^M \Rightarrow X + u\mathbb{1} \in A$.

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At Terminal Time.

- $K_T \colon \Omega \to \mathcal{P}\left(\mathbb{R}^d\right)$ (measurable) solvency cone mapping

K_T**-compatible:** $X \in A$, $X' \in K_T$ a.s. $\Rightarrow X + X' \in A$.

One-to-one properties for M-translative functions R and $A \subseteq L_d^p$:

$$A_R = \left\{X \in L_d^p \colon 0 \in R(X)\right\}, \quad R_A(X) = \{u \in M \colon X + u\mathbb{1} \in A\}$$

	R	A
finite at zero	$R(0) \neq \emptyset$ $R(0) \neq M$	$M\mathbb{1} \cap A \neq \emptyset$ $M\mathbb{1} \cap (L_d^p \backslash A) \neq \emptyset$
market-compatible	$L_d^p(K_T)$-monotone $R(X) = R(X) + K_0^M$	$A + L_d^p(K_T) \subseteq A$ $A + K_0^M \mathbb{1} \subseteq A$
	convex positively homogeneous subadditive sublinear closed images closed graph	convex cone $A + A \subseteq A$ convex cone directionally closed closed

$\|\blacktriangleright$ **Duality.**

Result. If a function $R\colon L_d^p \to \mathcal{P}(M)$ is convex (closed), then $R(X)$ is convex (closed) for all $X \in L_d^p$. A closed convex K_I-compatible risk measure R maps into

$$\mathbb{G}(M) = \left\{D \subseteq \mathbb{R}^d\colon D = \operatorname{cl}\operatorname{co}\left(D + K_I^M\right)\right\}.$$

Here: convexity, closedness in terms of the graph

$$\operatorname{gr} R = \left\{(X, u) \in L_d^p \times M\colon u \in R(X)\right\}.$$

Dual representation theorem. $R\colon L_d^p \to \mathbb{G}(M)$ is a closed convex market-compatible risk measure if and only if there is a penalty function $-\alpha\colon \mathcal{W}^q \to \mathbb{G}(M)$ such that for all $X \in L_d^p$

$$R(X) = \bigcap_{(Q,w)\in\mathcal{W}^q} \left\{-\alpha(Q,w) + \left(E^Q[-X] + G(w)\right) \cap M\right\}.$$

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In this case,

$$-\alpha(Q,w) \subseteq \text{cl} \bigcup_{X'\in A_R} \left(E^Q\left[X'\right] + G(w)\right) \cap M$$

with $G(w) = \left\{x \in \mathbb{R}^d\colon 0 \le w^T x\right\}$ and

$$\mathcal{W}^q = \left\{(Q,w) \in \mathcal{M}^P_{1,d} \times \left(K^+_I \backslash M^\perp + M^\perp\right) : \text{diag}(w)\frac{dQ}{dP} \in L^q_d\left(K^+_T\right)\right\}.$$

A note about the proof. Fenchel-Moreau theorem for set-valued functions, Hamel 09.

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A note about dual variables. Assume $M = \mathbb{R}^d$. Then

$$\mathcal{W}^q = \left\{(Q, w) \in \mathcal{M}_{1,d}^P \times K_I^+ \setminus \{0\} : \text{diag}\,(w)\,\frac{dQ}{dP} \in L_d^q\left(K_T^+\right)\right\}.$$

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Transformation of variables. $Y = \text{diag}\,(w) \frac{dQ}{dP}$, $E[Y] = w \in K_I^+ \backslash \{0\}$.

This gives: The pair (Y, w) is a consistent pricing process for the one-period market $(K_I, K_T = K_T(\omega))$.

The coherent case. R additionally positively homogeneous:

$$\forall X \in L_d^p \colon R(X) = \bigcap_{(Q,w) \in \mathcal{W}_R^q} \left(E^Q[-X] + G(w) \right) \cap M.$$

with

$$\mathcal{W}_R^q \subseteq \left\{ (Q, w) \in \mathcal{M}_{1,d}^P \times \left(K_I^+ \backslash M^\perp + M^\perp \right) : \text{diag}(w) \frac{dQ}{dP} \in A_R^+ \right\}.$$

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The coherent case with $M = \mathbb{R}^d$.

$$\mathcal{W}_R^q \subseteq \left\{(Q,w) \in \mathcal{M}_{1,d}^P \times K_I^+ \backslash \{0\} : \operatorname{diag}(w)\frac{dQ}{dP} \in A_R^+\right\}.$$

‖▶ **Super-hedging price.**

- $\Theta = \{t_0 = 0, t_1, \ldots, t_N = T\}$, $\left(\Omega, (\mathcal{F}_t)_{t\in\Theta}, \mathcal{F}, P\right)$, $\mathcal{F}_T = \mathcal{F}$;
- $(K_t(\omega))_{t\in\Theta}$ cone-valued process with $\mathbb{R}^d_+ \subseteq K_t(\omega) \subseteq \mathbb{R}^d$, $K_t(\omega) \neq \mathbb{R}^d$ closed convex P-a.s. for all $t \in \Theta$;
- Self–financing portfolio process: adapted $\mathbb{R}^d$–valued process $V = (V_t)_{t\in\Theta}$ with $(V_{t_{-1}} = 0)$

$$V_{t_n} - V_{t_{n-1}} \in -K_{t_n} \quad \text{a.s., } n = 1, \ldots, N-1$$

- The attainable set

$$A_t = \{V_t \colon V \text{ is a self-financing portfolio process}\}, \ t \in \Theta$$

is a convex cone in $L^0\left(\Omega, \mathcal{F}_t, P; \mathbb{R}^d\right)$.

Result. Assume (NA^r). Then $X \mapsto \left\{u \in \mathbb{R}^d \colon X + u\mathbb{1} \in -A_T\right\}$ is a closed coherent market-compatible risk measure with $K_I = K_0$.

Note. $-A_T = K_0\mathbb{1} + L^0_d\left(K_{t_1}\right) + \ldots + L^0_d\left(K_T\right)$.

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Note. $-A_T = K_0\mathbb{1} + L_d^0\left(K_{t_1}\right) + \ldots + L_d^0\left(K_T\right)$.

Super-hedging theorem. $X \in L_d^1$, $v \in \mathbb{R}^d$

$$X - v\mathbb{1} \in A_T \quad \Leftrightarrow \quad \forall Z \in SCPP \colon E\left[X^T Z_T\right] \leq v^T Z_0.$$

This produces the dual representation of the super-hedging price in terms of (Q, w) via the following transformation of variables.

Transformation of variables. Set $w = E[Z_T] = Z_0 \in K_0^+ \setminus \{0\}$ and

$$\frac{dQ_i}{dP} = \frac{1}{w_i} (Z_T)_i \quad \text{if } w_i > 0,$$

and choose $\frac{dQ_i}{dP}$ as density in L_+^∞ if $w_i = 0$. Then

$$(Q, w) \in \mathcal{M}_{1,d}^P \times K_0^+ \setminus \{0\}$$

$$E\left[\text{diag}(w) \frac{dQ}{dP} | \mathcal{F}_t\right] \in L_d^p \left(K_t^+\right), \; t \in \Theta$$

In particular, $\text{diag}(w) \frac{dQ}{dP} \in K_T^+$ P-a.s. Moreover, $E\left[X^T Z_T\right] = w^T E^Q[X]$ and $Z_0^T u = w^T u$, hence the following result.

Result. $X \in L^1_d$. Then,

$$R_{-A_T}(-X) = \bigcap_{(Q,w) \in \mathcal{W}^{\infty}_{SCPP}} \left(E^Q[X] + G(w) \right)$$

with

$$\mathcal{W}^{\infty}_{SCPP} = \Big\{ (Q,w) \in \mathcal{M}^P_{1,d} \times K_0^+ \backslash \{0\} : \\ \forall t \in \Theta : E\left[\mathrm{diag}(w) \frac{dQ}{dP} | \mathcal{F}_t \right] \in L^p_d\left(K_t^+\right) \Big\}.$$

Summary. Set-valued duality covers both super-hedging theorems and dual representation of risk measures in conical market models.

$\|\blacktriangleright$ **AV@R.**

Recall (from dual representation theorem for $q = \infty$)

$$\mathcal{W}^{\infty} = \left\{ (Q, w) \in \mathcal{M}_{1,d}^{P} \times \left(K_I^{+} \backslash M^{\perp} + M^{\perp} \right) : \text{diag}\,(w) \, \frac{dQ}{dP} \in L_d^{\infty} \left(K_T^{+} \right) \right\}.$$

If $\alpha \in (0, 1]^d$,

$$\mathcal{W}_{\alpha}^{\infty} = \left\{ (Q, w) \in \mathcal{W}^{\infty} : \text{diag}\,(w) \left(\alpha \mathbb{1} - \frac{dQ}{dP} \right) \in L_d^{\infty} \left(K_T^{+} \right) \right\}$$

then

$$AV@R_{\alpha}(X) = \bigcap_{(Q,w) \in \mathcal{W}_{\alpha}^{\infty}} \left(E^{Q}[-X] + G(w) \right) \cap M$$

defines a market-compatible sublinear (coherent) risk measure on L_d^1.

Note. This is a "dual-way" definition! And a new one, by the way.

Questions.

1. Computing values $AV@R_\alpha(X)$?

2. Minimizing $AV@R_\alpha(X)$ over $X \in C \subseteq L^1_d$?

$\|$ ▶ Computing the value $AV@R_\alpha(X)$.

Fact 1.

$$AV@R_\alpha(X) = \bigcap_{(Q,w)\in\mathcal{W}_\alpha^\infty} \left(E^Q[-X] + G(w)\right) \cap M$$
$$= \bigcap_{(Y,v)\in\mathcal{Y}_\alpha} \left\{u \in M \colon E\left[-Y^T X\right] \le v^T u\right\}$$

with

$$\mathcal{Y}_\alpha = \left\{ (Y,v) \in L_d^\infty \times M\backslash\{0\} : \begin{array}{l} v \in \left(E[Y] + M^\perp\right) \cap \left(K_I^+ + M^\perp\right) \\ Y \in K_T^+ \backslash \{0\} \\ \text{diag}(\alpha)E[Y] - Y \in K_T^+ \end{array} \right\}.$$

Note. Linear in (Y, v).

$\|$▶ **Computing the value $AV@R_\alpha(X)$.**

Fact 2. If $M = \mathbb{R}^d$ this simplifies to

$$AV@R_\alpha(X) = \bigcap_{(Q,w)\in\mathcal{W}_\alpha^\infty} \left(E^Q[-X] + G(w)\right)$$
$$= \bigcap_{(Y,v)\in\mathcal{Y}_\alpha^d} \left\{u \in \mathbb{R}^d \colon E\left[-Y^T X\right] \leq v^T u\right\}$$

with

$$\mathcal{Y}_\alpha^d = \left\{(Y, v) \in L_d^\infty\left(K_T^+\right) \times K_I^+ \backslash \{0\} \colon \right.$$
$$\left. v = E[Y],\ \text{diag}(\alpha)\, v - Y \in L_d^\infty\left(K_T^+\right)\right\}.$$

$\|\blacktriangleright$ **Computing the value $AV@R_\alpha(X)$.**

Further assumptions.

- $|\Omega|$, $M = \mathbb{R}^d$,
- K_I is spanned by $h^1, \ldots, h^{J_I}$
- $K_T(\omega)$ is spanned by $k^1(\omega), \ldots, k^{J_T}(\omega)$

Note.

- $Y \in K_T^+$ $\leftrightsquigarrow$ $Y \geq 0$
- $\operatorname{diag}(\alpha)\, v - Y \in K_T^+$ P-a.s. $\leftrightsquigarrow$ $Y \leq \operatorname{diag}(\alpha)\, v$
- $\bigcap$ $\leftrightsquigarrow$ sup
- $X \mapsto \left\{u \in \mathbb{R}^d \colon E\left[-Y^T X\right] \leq v^T u\right\}$ "almost linear"

$\|$ ▶ Computing the value $AV@R_\alpha(X)$.

Analyzing the constraints.

- $Y \in K_T^+$: $y_{in} = Y_i(\omega_n)$, $i = 1, \ldots, d$, $n = 1, \ldots, N$

$$\forall j = 1, \ldots, J_T, \forall n = 1, \ldots, N\colon \sum_{i=1}^{d} y_{in} k_{in}^{j} \geq 0$$

with $k_{in}^{j} = k_i^j(\omega_n)$. This gives NJ_T linear inequality constraints.

$\|\blacktriangleright$ Computing the value $AV@R_\alpha(X)$.

Analyzing the constraints.

- diag $(\alpha)\, v - Y \in K_T^+$:

$$\forall j = 1, \ldots, J_T, \forall n = 1, \ldots, N \colon \sum_{i=1}^{d} y_{in} k_{in}^j \le \sum_{i=1}^{d} \alpha_i k_{in}^j v_i.$$

  This gives another NJ_T linear inequality constraints.

$\|\blacktriangleright$ **Computing the value $AV@R_\alpha(X)$.**

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- $\text{diag}(\alpha)\, v - Y \in K_T^+$:

$$\forall j = 1, \ldots, J_T, \forall n = 1, \ldots, N\colon \sum_{i=1}^{d} y_{in} k_{in}^{j} \leq \sum_{i=1}^{d} \alpha_i k_{in}^{j} v_i.$$

  This gives another NJ_T linear inequality constraints.

- $v = E[Y]$:

$$\forall i = 1, \ldots, d\colon \sum_{n=1}^{N} p_n y_{in} = v_i.$$

  This gives d linear equations.

$\|\blacktriangleright$ **Computing the value $AV@R_{\alpha}(X)$.**

Analyzing the objective.

- $\left\{u \in \mathbb{R}^d \colon E\left[-Y^T X\right] \leq v^T u\right\}$:

$$E\left[-Y^T X\right] = -\sum_{i=1}^{d} \sum_{n=1}^{N} p_n x_{in} y_{in},$$

therefore the objective becomes

$$S_{(\widehat{D}\widehat{y}, -v)}(-\widehat{x}) = \left\{u \in \mathbb{R}^d \colon -\widehat{x}^T \widehat{D}\widehat{y} \leq v^T u\right\}.$$

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Altogether.

$$AV@R_\alpha(X) = \bigcap \left\{S_{(\hat{D}\hat{y},-v)}(-\hat{x}) : A_1^T \hat{y} \leq -C_1^T v,\ A_2^T \hat{y} = -C_2^T v,\ v \in K_I^+\right\}$$

with suitable matrices A_1, A_2, C_1, C_2, $\hat{D}$, $\hat{x}, \hat{y}$.

Reference. Yankova 10, JP, P.U.

$\|$ ▶ Computing the value $AV@R_\alpha(X)$.

Constructing the primal.

The problem

$$\bigcap \left\{ S_{(\hat{D}\hat{y}, -v)}(-\hat{x}) : A_1^T \hat{y} \leq -C_1^T v,\ A_2^T \hat{y} = -C_2^T v,\ v \in K_I^+ \right\}$$

is the set-valued dual of the following set-valued linear program

$$\inf_{\mathbb{G}(\mathbb{R}^d)} \left\{ C_1 x^1 + C_2 x^2 \colon A_1 x^1 + A_2 x^2 = -\hat{x},\ x^1 \geq 0 \right\}.$$

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Interpretation as vector optimization problem. Look for minimal points of

$$\left\{ \mathrm{diag}\,(\alpha)\, E\,[Z] - z \colon Z \in L_d^q(K_T)\,,\ Z - z\mathbb{1} + X \in L_d^q(K_T)\,,\ z \in \mathbb{R}^d \right\}$$

with respect to the order relation in $\mathbb{R}^d$ generated by K_I.

Reference. Hamel 10+

‖▶ Computing the value $AV@R_\alpha(X)$.

Under the additional assumptions and $M = \mathbb{R}^d$

$$
\begin{aligned}
& AV@R_\alpha(X) \\
& = \left\{\mathrm{diag}(\alpha)\, E[Z] - z \colon Z \in L^q_d(K_T),\ Z - z\mathbb{1} + X \in L^q_d(K_T),\ z \in \mathbb{R}^d\right\} \\
& = \bigcap_{(Y,v)\in\mathcal{Y}^d_\alpha} \left\{u \in \mathbb{R}^d \colon E\left[-Y^T X\right] \le v^T u\right\}
\end{aligned}
$$

with

$$
\mathcal{Y}^d_\alpha = \left\{(Y, v) \in L^\infty_d\left(K^+_T\right) \times K^+_I \backslash \{0\} : v = E[Y],\ \mathrm{diag}(\alpha)\, v - Y \in K^+_T\right\}
$$

$\Vert$ ▶ Computing the value $AV@R_\alpha(X)$.

Under the additional assumptions and $M = \mathbb{R}^d$

$$
\begin{aligned}
&AV@R_\alpha(X)\\
&= \left\{\text{diag}(\alpha) E[Z] - z \colon Z \in L^q_d(K_T),\ Z - z\mathbb{1} + X \in L^q_d(K_T),\ z \in \mathbb{R}^d\right\}\\
&= \bigcap_{(Y,v)\in\mathcal{Y}^d_\alpha} \left\{u \in \mathbb{R}^d \colon E\left[-Y^T X\right] \le v^T u\right\}
\end{aligned}
$$

with

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\mathcal{Y}^d_\alpha = \left\{(Y, v) \in L^\infty_d\left(K^+_T\right) \times K^+_I \backslash \{0\} : v = E[Y],\ \text{diag}(\alpha) v - Y \in K^+_T\right\}
$$

Good news. There are already efficient algorithms for such (vector) problems (Benson 1998, Ehrgott/Löhne/Shao 2007).

$\|\blacktriangleright$ **Computing the value** $AV@R_{\alpha}(X)$.

Under the additional assumptions and $M = \mathbb{R}^d$

$$
\begin{aligned}
&AV@R_{\alpha}(X) \\
&= \left\{ \text{diag}(\alpha) E[Z] - z \colon Z \in L^q_d(K_T),\ Z - z1\!\!1 + X \in L^q_d(K_T),\ z \in \mathbb{R}^d \right\} \\
&= \bigcap_{(Y,v) \in \mathcal{Y}^d_\alpha} \left\{ u \in \mathbb{R}^d \colon E\left[-Y^T X\right] \le v^T u \right\}
\end{aligned}
$$

with

$$\mathcal{Y}^d_\alpha = \left\{ (Y, v) \in L^\infty_d\left(K^+_T\right) \times K^+_I \backslash \{0\} : v = E[Y],\ \text{diag}(\alpha) v - Y \in K^+_T \right\}$$

Good news. There are already efficient algorithms for such (vector) problems (Benson 1998, Ehrgott/Löhne/Shao 2007).

Summary. Computation of values of a set-valued risk measure is a vector/set optimization problem. Set-valued duality provides tools.

$\|\blacktriangleright$ **What's next?**

- Computing super-hedging prices and values of AV@R.

- Set-valued optimization problems for set-valued risk measures.

- Law invariance of set-valued risk measures.

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Thanks for coming.