Risk Measures in non-dominated Models

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Purpose: Study Risk Measures taking into account the model uncertainty in mathematical finance.

Plan

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   - Model Uncertainty
   - Fundamental topological Properties
2. Risk Measures on $\mathcal{L}_p(c)$
   - Definitions
   - Representation of Risk Measure on $\mathcal{L}_p(c)$
   - Examples
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Non-dominated Models

Model Uncertainty

Motivations

- Lots of probability models in mathematical finance to describe prices of assets
- For each model, problems of calibration
- It is natural to consider not completely specified models
- Example of framework taking into account model uncertainty : UVM (uncer tain volatility model). It is a generalization of the Black-Scholes model : the volatility process is not known but it is assumed to lie in a fixed interval.

In order to take into account model uncertainty, we consider that knowledge is not represented by one probability measure, but by a set of probability measures $\mathcal{P}$.

Framework

Framework

- d risky assets.
- Let $\Omega = C_0([0;T], \mathbb{R}^d)$be the space of continuous functions defined on $[0, T]$ with values in $\mathbb{R}^d$, vanishing in 0, endowed with the uniform convergence norm, $(\mathcal{F}_t)$ the canonical filtration.
- $(B_t)_{t\in[0,T]}$ the coordinates process, and for all $i \in \{1, \dots, d\}$, $(B_{i,t})_{t\in[0;T]}$ the process of the i^{th} coordinate.
- For all $i \in \{1, \dots, d\}$, we consider $\underline{\mu}_i$ and μ_i finite deterministic measures on $[0, T]$. We assume that for all i, μ_i and $\underline{\mu}_i$ are Hölder-continuous.

Hypothesis

Definitions

- A probability measure P is an orthogonal martingale law if the coordinate process is a martingale with respect to $(\mathcal{F}_t)$ under P, and if the martingales $(B_i)_{1\leqslant i\leqslant d}$ are orthogonal, i.e.

$$\forall i \neq j, \forall t \in [0,T], \ < B_i, B_j >_t^P = 0 \ \ P \ a.s.$$

- An orthogonal martingale law satisfies the property $H(\underline{\mu}, \mu)$ if

$$\forall i \in \{1,\dots,d\}, \quad d\underline{\mu}_{i,t} \leqslant d < B_i >_t^P \leqslant d\mu_{i,t}.$$

Let $\mathcal{P}_0$ be the set of orthogonal martingale law satisfying the property $H(\underline{\mu}, \mu)$.(the case where $d\underline{\mu}_{i,t} = \underline{\sigma}_i^2 dt$ and $d\mu_{i,t} = \sigma_i^2 dt$ for some constants $\underline{\sigma}_i$ and σ_i, correspond to the case where the volatility of the i^{th} asset belongs to a fixed interval)

$$\mathcal{P} = \left\{ Q; \exists P_0 \in \mathcal{P}_0, \frac{dQ}{dP_0} = \exp\left(\int_0^T \lambda_s dB_s - \frac{1}{2}\int_0^T |\lambda_s|^2 ds \right) \text{ avec } |\lambda| \leqslant C \right\}$$

Difficulties and capacities

Difficulties linked to non-dominated models

- Definition of a negligible set?
- Definition of ”almost surely”?

Definitions

We define a regular capacity on $C_b(\Omega)$ with:

$$\forall f \in C_b(\Omega),\ c_p(f) = \sup_{P \in \mathcal{P}_0} E_P[|f|^p]^{\frac{1}{p}}$$

Then, we consider the Lebesgue extension to all functions.

Definitions

- A set A is said to be polar if $c_p(\mathbf{1}_A) = 0$
- A property is said to hold ”quasi-surely” if it holds outside a polar set.

Fundamental topological Properties

Let $\mathcal{L}_p(c)$ be the topological completion of $C_b(\Omega)$ with respect to the semi-norm c_p.

Theorem of D. Feyel/A. De la Pradelle, dual of $\mathcal{L}_p(c)$

If T is a non negative linear form on $\mathcal{L}_p(c)$ then T is continuous and there exists a non negative finite measure λ which does not charge polar sets such that

$$\forall f \in \mathcal{L}_p(c),\ T(f) = \int_\Omega f(x)\lambda(dx)$$

Théorème

$\mathcal{P}$ is weakly compact and convex.

Risk Measures on $\mathcal{L}_p(c)$

Definition of Risk Measures on $\mathcal{L}_p(c)$

Définition

A map $\rho : \Pi \rightarrow \mathbb{R}$ is said to be a monetary risk measure if it satisfies the following assertions for all $X, Y \in \Pi$:

- Monotonicity: If $X \leqslant Y$ quasi-everywhere, then $\rho(X) \geqslant \rho(Y)$
- Translation Invariance: $\forall m \in \mathbb{R}$, $\rho(X + m) = \rho(X) - m$

A monetary risk measure is said to be convex if it satisfies the following assertion:

- Convexity: $\forall \lambda \in [0, 1]$, $\forall X, Y \in \Pi$, $\rho(\lambda X + (1 - \lambda) Y) \leqslant \lambda \rho(X) + (1 - \lambda) \rho(Y)$

A convex risk measure is said to be a coherent measure of risk if it satisfies the following additional assertion:

- Positive Homogeneity: $\forall \lambda \geqslant 0$, $\forall X \in \Pi$, $\rho(\lambda X) = \lambda \rho(X)$

Representation of Coherent Risk Measure on $\mathcal{L}_p(c)$

Theorem

Let ρ be a coherent risk measure on $\mathcal{L}_p(c)$. Then, ρ is continuous and there exists a set $\mathcal{Q}$ of probability measures on Ω which do not charge polar sets and such that

$$\forall X \in \mathcal{L}_p(c),\ \rho(X) = \sup_{P \in \mathcal{Q}} E_P[-X]$$

Moreover, $\mathcal{Q}$ can be chosen convex and such that the supremum above is attained.

Representation of Convex Risk Measure on $\mathcal{L}_p(c)$

Theorem

Let ρ be a convex risk measure on $\mathcal{L}_p(c)$. Then, ρ admits the following representation:

$$\forall X \in \mathcal{L}_p(c),\ \rho(X) = \sup_{Q \in \mathcal{P}'} E_Q[-X] - \alpha_{min}(Q)$$

where

$$\alpha_{min}(Q) = \sup_{X \in \mathcal{L}_p(c)} (E_Q[-X] - \rho(X)) = \sup_{X \in \mathcal{A}_\rho} E_Q[-X]$$

and $\mathcal{P}'$ the set of probability measures which do not charge polar sets and belonging to $(\mathcal{L}_p(c))^\star$.
Moreover, for all $X \in \mathcal{L}_p(c)$, there exists a probability measure $P_X \in \mathcal{P}'$ such that:

$$\rho(X) = E_{P_X}[-X] - \alpha_{min}(P_X)$$

And, α_{min} is the minimal penality function, meaning that if α is another

A Monetary Risk Measure

Definition

Let $\alpha \in (0,1)$. We define

$$\forall X \in \mathcal{L}_p(c),\ V@R_\alpha(X) = -\inf\{x \in \mathbb{R},\ \exists P \in \mathcal{P}, P(X \leqslant x) > \alpha\}.$$

Properties

Let $p \geqslant 1$, then for all $X \in \mathcal{L}_p(c)$, $V@R_\alpha(X)$ is finite and

$$V@R_\alpha(X) = \sup_{P \in \mathcal{P}} V@R_\alpha^P(X) = \sup_{P \in \mathcal{P}} \left(-\inf\{x \in \mathbb{R},\ P(X \leqslant x) > \alpha\}\right)$$

Moreover, $V@R_\alpha$ is a monetary risk measure on $\mathcal{L}_p(c)$

A Coherent Risk Measure

Définition(Föllmer/Shied)

Let $\lambda \in (0,1)$. We put

$$\forall X \in L^{\infty}(P),\ AV@R_{\lambda}^{P}(X) = \frac{1}{\lambda}\int_0^{\lambda} V@R_{\alpha}^{P}(X)\,d\alpha$$

Définition

Let $\lambda \in (0,1)$ and $p > 1$. We put

$$\forall X \in \mathcal{L}_p(c),\ AV@R_{\lambda}(X) = \sup_{P\in\mathcal{P}}\left(\frac{1}{\lambda}\int_0^{\lambda} V@R_{\alpha}^{P}(X)\,d\alpha\right)$$

$$\forall X \in L^{\infty}(P)\cap\mathcal{L}_p(c),\ AV@R_{\lambda}(X) = \sup_{P\in\mathcal{P}} AV@R_{\lambda}^{P}(X)$$

A Coherent Risk Measure

Theorem

For all $p > 1$ and $\lambda \in (0,1)$, $AV@R_\lambda$ is a coherent risk measure on $\mathcal{L}_p(c)$ which admits the following representation:

$$(1) \qquad \forall X \in \mathcal{L}_p(c),\; AV@R_\lambda(X) = \sup_{Q \in \mathcal{Q}_\lambda} E_Q[-X]$$

where $\mathcal{Q}_\lambda = \left\{Q \in \mathcal{P}';\ \exists P \in \mathcal{P}\ s.t.\ Q \ll P\ and\ \frac{dQ}{dP} \leqslant \frac{1}{\lambda}\ P - a.s\right\}$.
Moreover, the supremum in (1) is attained.

A Convex Risk Measure

Let $l : \mathbb{R} \to \mathbb{R}$ be an increasing, convex loss function not identically constant and such that there exists a constant $M > 0$ satisfying

$$\forall x \in \mathbb{R}, \quad |l(x)| \leqslant M\left(1 + |x|^p\right)$$

For example, we can take: $\forall x \in \mathbb{R},\ l(x) = \left(x^+\right)^p$. Assume that x_0 belongs to the interior of $l(\mathbb{R})$ and let $\mathcal{A}$ be the set of acceptable positions :

$$\mathcal{A} = \{X \in \mathcal{L}_p(c),\ \forall P \in \mathcal{P},\ E_P\left[l(-X)\right] \leqslant x_0\}$$

Thus, a financial position is considered to be acceptable if whatever the scenario that happens, whatever the model (i.e. for all $P \in \mathcal{P}$), the mean loss is smaller than x_0.

Shortfall risk

$$\begin{aligned} \rho_{l,x_0}(X) &= \inf\{m \in \mathbb{R};\ m + X \in \mathcal{A}\} \\ &= \inf\{m \in \mathbb{R};\ \forall P \in \mathcal{P},\ E_P\left[l(-m - X)\right] \leqslant x_0\} \end{aligned}$$

ρ_{l,x_0} is a convex risk measure on $\mathcal{L}_p(c)$.

Perspectives

Difficultés et perspectives:

- Link with G-expectations.
- Definition and study of dynamic risk measure in a framework taking into account model uncertainty.