Numerical valuation for option pricing under jump-diffusion models by finite differences

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We assume that the stock price process S_t in a risk-neutral world follows an exponential jump-diffusion model

$$dS_t/S_{t-} = (r - \lambda\zeta)dt + \sigma dW_t + \eta dN_t, \tag{1}$$

where

r : the riskfree interest rate,

σ : the volatility,

W_t : the Wiener process,

N_t : the Poisson process with intensity λ,

η : a random variable of jump size from S_{t-} to $(\eta + 1)S_{t-}$,

ζ : the expectation $\mathbb{E}[\eta]$ of the random variable η.

Example (Jump-diffusion model)

(1) Merton model

$$\ln(\eta+1) \sim N(\mu_J, \sigma_J^2). \quad (2)$$

(2) Kou model

$$f(x) = p\lambda_+ e^{-\lambda_+ x} 1_{x \geq 0} + (1-p)\lambda_- e^{\lambda_- x} 1_{x<0}, \quad (3)$$

where $f(x)$ is a density function of $\ln(\eta+1)$.

Under exponential jump-diffusion model, the price of a European call option $C(t,S)$ satisfies the PIDE below.

$$\frac{\partial C}{\partial t}(t,S) + \frac{\sigma^2 S^2}{2}\frac{\partial^2 C}{\partial S^2}(t,S) + rS\frac{\partial C}{\partial S}(t,S) - rC(t,S) \\ + \int_{\mathbb{R}} \left[C(t,Se^x) - C(t,S) - S(e^x-1)\frac{\partial C}{\partial S}(t,S) \right] \nu(dx) = 0$$

on $[0,T) \times (0,\infty)$ with the terminal condition

$$C(T,S) = (S-K)^+ \quad \text{for all } S > 0,$$

where K is a strike price.

Let

$$\tau = T - t, \quad x = \ln(S/S_0).$$

By the change of variables, $u(\tau, x) = C(T - \tau, S_0 e^x)$ satisfies

$$\frac{\partial u}{\partial \tau}(\tau, x) = \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2}(\tau, x) + (r - \frac{\sigma^2}{2} - \lambda\zeta)\frac{\partial u}{\partial x}(\tau, x) \\ -(r + \lambda)u(\tau, x) + \lambda \int_{\mathbb{R}} u(\tau, z) f(z - x) dz \quad (4)$$

on $(0, T] \times (-\infty, \infty)$ with the initial condition

$$u(0, x) = (S_0 e^x - K)^+ \quad \text{for all } x \in (-\infty, \infty), \quad (5)$$

where $\zeta = \int_{\mathbb{R}}(e^x - 1)f(x)dx$ with the distribution function of jumps $f(x)$ and λ is the intensity of jumps.

In the sense of the viscosity solution,

- Briani, Chioma, and Natalini (2004)
  - An explicit difference method.
- Cont and Voltchkova (2005)
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As using an iterative method,

- d'Halluin, Forsyth, and Vetzal (2005)
  - An implicit method of the Crank-Nicolson type.
- Almendral and Oosterlee (2005)
  - A backward differentiation formula (BDF2).

We shall construct a numerical method with finite differences to solve the following initial-valued PIDE

$$\frac{\partial u}{\partial \tau}(\tau, x) = \mathcal{L}u(\tau, x) \quad \text{on } (0, T] \times \mathbb{R}, \tag{6}$$

$$u(0, x) = (S_0 e^x - K)^+, \tag{7}$$

where

$$\begin{aligned}
\mathcal{L}u(\tau, x) =& \mathcal{D}u(\tau, x) + \mathcal{I}u(\tau, x) - (r + \lambda)u(\tau, x), \\
\mathcal{D}u(\tau, x) =& \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2}(\tau, x) + (r - \frac{\sigma^2}{2} - \lambda\zeta)\frac{\partial u}{\partial x}(\tau, x), \\
\mathcal{I}u(\tau, x) =& \lambda \int_{\mathbb{R}} u(\tau, z) f(z - x) dz.
\end{aligned}$$

At first, we have to restrict the domain $\mathbb{R}$ of the space variable to a bounded interval. The asymptotic behavior of the price of a European call option is described by

$$\lim_{x\to-\infty} u(\tau,x) = 0, \quad \lim_{x\to\infty} u(\tau,x) = S_0 e^x - Ke^{-r\tau}. \qquad (8)$$

So, there exists an interval $\Omega := [-X, X], X > 0$ such that we can divide the integral term into two parts

$$\int_{\mathbb{R}} u(\tau,z) f(z-x) dz = \int_{\Omega} u(\tau,z) f(z-x) dz + \int_{\mathbb{R}\setminus\Omega} u(\tau,z) f(z-x) dz$$

Let us define $R(\tau, x, X)$ by $R(\tau, x, X) = \int_{\mathbb{R}\setminus\Omega} u(\tau, z) f(z-x) dz$.
In the case of Merton model

$$R(\tau, x, X) \\ = S_0 e^{x+\mu_J+\frac{\sigma_J^2}{2}} \Phi\left(\frac{x - X + \mu_J + \sigma_J^2}{\sigma_J}\right) - Ke^{-r\tau}\Phi\left(\frac{x - X + \mu_J}{\sigma_J}\right),$$

where

$$\Phi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{x^2}{2}} dx.$$

In the case of Kou model

$$R(\tau, x, X) = S_0 \frac{p\lambda_+}{\lambda_+ - 1} e^{\lambda_+ x - (\lambda_+ - 1)X} - Kpe^{-r\tau - \lambda_+(X - x)}.$$

On the truncated domain $[0, T] \times [-X, X]$, let $\Delta\tau = T/N$ and $\Delta x = 2X/M$ for $M, N > 0$. And let $\tau_n = n\Delta\tau$ for $n = 0, 1, \ldots, N$ and $x_m = -X + m\Delta x$ for $m = 0, 1, \ldots, M$. Let $u_m^n = u(\tau_n, x_m)$ and $f_{m,j} = f(x_j - x_m)$.

$$\int_\Omega u(\tau_n, z) f(z - x_m) dz \approx \frac{\Delta x}{2} \left(u_0^n f_{m,0} + 2 \sum_{j=1}^{M-1} u_j^n f_{m,j} + u_M^n f_{m,M} \right).$$

$$\mathcal{D}u_m^n \approx \mathcal{D}_\Delta\left(\frac{u_m^{n+1}+u_m^{n-1}}{2}\right), \quad \mathcal{I}u_m^n \approx \mathcal{I}_\Delta u_m^n, \quad \mathcal{L}u_m^n \approx \mathcal{L}_\Delta u_m^n,$$

where

$$\mathcal{D}_\Delta u_m^n = \frac{\sigma^2}{2}\frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{\Delta x^2} + \left(r - \frac{\sigma^2}{2} - \lambda\zeta\right)\frac{u_{m+1}^n - u_{m-1}^n}{2\Delta x},$$

$$\mathcal{I}_\Delta u_m^n = \frac{\lambda\Delta x}{2}\left(u_0^n f_{m,0} + 2\sum_{j=1}^{M-1} u_j^n f_{m,j} + u_M^n f_{m,M}\right) + \lambda R(\tau_n, x_m, X),$$

$$\mathcal{L}_\Delta u_m^n = \begin{cases} \mathcal{D}_\Delta u_m^n + \mathcal{I}_\Delta u_m^n - (r+\lambda)u_m^n & \text{for } n = 0, \\ \mathcal{D}_\Delta\left(\frac{u_m^{n+1}+u_m^{n-1}}{2}\right) + \mathcal{I}_\Delta u_m^n - (r+\lambda)u_m^n & \text{for } n \geq 1. \end{cases}$$

Algorithm of the implicit method with three time levels

Initial condition:

$U_m^0 = \max(S_0 e^{x_m} - K, 0) \quad \text{for} \;\; 0 \le m \le M,$

Boundary condition: for $m = 0, M$ and for $1 \le n \le N$

$U_m^n = \max(0, S_0 e^{x_m} - K e^{-r\tau_n}),$

(S1) For $n = 0$ and for $1 \le m \le M - 1$

$\frac{U_m^{n+1} - U_m^n}{\Delta\tau} = \mathcal{D}_\Delta U_m^n + \mathcal{I}_\Delta U_m^n - (r + \lambda) U_m^n,$

(S2) For $1 \le n \le N - 1$ and for $1 \le m \le M - 1$

$\frac{U_m^{n+1} - U_m^{n-1}}{2\Delta\tau} = \mathcal{D}_\Delta \left(\frac{U_m^{n+1} + U_m^{n-1}}{2} \right) + \mathcal{I}_\Delta U_m^n - (r + \lambda) U_m^n.$

Theorem (Consistency)

Let $v \in C^\infty((0,T] \times \mathbb{R})$ satisfy the asymptotic behavior (8). If $\Delta\tau$ and Δx are sufficiently small, Then for any $\epsilon > 0$ there exists a truncated interval $[-X, X]$ such that

$$\frac{\partial v}{\partial \tau}(\tau_n, x_m) - \mathcal{L}v(\tau_n, x_m) - \left(\frac{v(\tau_{n+1}, x_m) - v(\tau_n, x_m)}{\Delta\tau} - \mathcal{L}_\Delta v(\tau_n, x_m) \right) = O(\Delta\tau + \Delta x^2 + \epsilon) \quad \text{for } n = 0, \tag{9}$$

$$\frac{\partial v}{\partial \tau}(\tau_n, x_m) - \mathcal{L}v(\tau_n, x_m) - \left(\frac{v(\tau_{n+1}, x_m) - v(\tau_{n-1}, x_m)}{2\Delta\tau} - \mathcal{L}_\Delta v(\tau_n, x_m) \right) = O(\Delta\tau^2 + \Delta x^2 + \epsilon) \quad \text{for } n \geq 1, \tag{10}$$

where $(\tau_n, x_m) \in (0,T] \times [-X, X]$.

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where $(\tau_n, x_m) \in (0,T] \times [-X, X]$.

Theorem (Stability)

*The finite difference method (**S1**)-(**S2**) is stable in the sense of the Von Neumann analysis if $\Delta\tau < \frac{1}{2(r+2\lambda)}$.*

We shall use a discrete vector norm $\|x\|_{\ell^2}$ defined by

$$\|x\|_{\ell^2} = \left(\Delta x \sum_j |x_j|^2 \right)^{1/2} .$$

Let ξ^n be the error vector on the n-th time level by

$$\xi_m^n = u_m^n - U_m^n \qquad \text{for } 1 \leq m \leq M-1,$$

where u is the unique solution of the initial-valued PIDE in (6)-(7) and U is the solution of the finite difference approximation in (**S1**)-(**S2**).

Lemma

Let $\{a_n\}_{n\geq 0}$ be a nonnegative sequence such that for $n \geq 2$

$$a_n \leq a_{n-2} + K\Delta\tau a_{n-1} + d,$$

where $\Delta\tau$, K, d are positive constants. If $a_0 = 0$, then for $n \geq 2$

$$a_n \leq (1 + K\Delta\tau)^{n-1} a_1 + d \sum_{j=0}^{n-2} (1 + K\Delta\tau)^j.$$

Theorem (Convergence)

If $\Delta\tau$ and Δx are sufficiently small, then there exists a positive constant K independent of $\Delta\tau$ and Δx such that for $1 \leq n \leq N$

$$\|\xi^n\|_{\ell^2} \leq K(\Delta\tau^2 + \Delta x^2 + \frac{1}{\Delta x^{3/2}}\epsilon). \tag{11}$$

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Corollary

Suppose that all hypotheses in Theorem above are satisfied. If the conditions of $\epsilon = O(\Delta x^{7/2})$ and $\Delta x = O(\Delta\tau)$ hold, then

$$\|\xi^n\|_{\ell^2} \leq K(\Delta\tau^2). \tag{12}$$

Example

Under Merton model, parameters used in the simulation were

$$\sigma = 0.15, \quad r = 0.05, \quad \sigma_J = 0.45, \quad \mu_J = -0.90,$$
$$\lambda = 0.10, \quad T = 0.25, \quad K = 100.$$

The order q of convergence rate was computed by

$$q = \log_2 \frac{\|U(\Delta\tau, \Delta x) - U(\Delta\tau/2, \Delta x/2)\|_{\ell^2}}{\|U(\Delta\tau/2, \Delta x/2) - U(\Delta\tau/4, \Delta x/4)\|_{\ell^2}}. \tag{13}$$

Table: Values of European call options obtained by the implicit method with three time levels under the Merton model. The reference values are 0.527638 at $S = 90$, 4.391246 at $S = 100$, and 12.643406 at $S = 110$. The truncated domain is $[-1.5, 1.5]$. N is the number of time steps and M is the number of space steps.

		$S = 90$		$S = 100$		$S = 110$	
N	M	Value	Error	Value	Error	Value	Error
25	128	0.525183	0.002455	4.355963	0.035283	12.635554	0.007852
50	256	0.527098	0.000540	4.382389	0.008857	12.641354	0.002052
100	512	0.527497	0.000141	4.389039	0.002207	12.642889	0.000517
200	1024	0.527602	0.000036	4.390695	0.000551	12.643277	0.000129
400	2048	0.527629	0.000009	4.391108	0.000138	12.643373	0.000033
800	4096	0.527636	0.000002	4.391211	0.000035	12.643398	0.000008

Table: The rate of ℓ^2-errors obtained by the implicit method with three time levels under the Merton model. The truncated domain is $[-1.5, 1.5]$. N is the number of time steps and M is the number of space steps. q is the rate of convergence defined by (13).

N	M	$\|U(\Delta\tau, \Delta x) - U(\Delta\tau/2, \Delta x/2)\|_{\ell^2}$	q
25	128		
		0.008401099831144	-
50	256		
		0.002125993690576	1.982
100	512		
		0.000530484616757	2.003
200	1024		
		0.000132558327520	2.001
400	2048		
		0.000033135663449	2.000
800	4096		

Example

Under Kou model, parameters used in the simulation were

$$\sigma = 0.15, \quad r = 0.05, \quad \lambda_+ = 3.0465, \quad \lambda_- = 3.0775,$$

$$p = 0.3445, \quad \lambda = 0.10, \quad T = 0.25, \quad K = 100.$$

Table: Values of European call options obtained by the implicit method with three time levels under the Kou model. The reference values are 0.672677 at $S = 90$, 3.973479 at $S = 100$, and 11.794583 at $S = 110$. The truncated domain is $[-1.5, 1.5]$. N is the number of time steps and M is the number of space steps.

		$S = 90$		$S = 100$		$S = 110$	
N	M	Value	Error	Value	Error	Value	Error
25	128	0.669157	0.003520	3.939036	0.034443	11.786790	0.007793
50	256	0.671823	0.000854	3.964816	0.008663	11.792574	0.002009
100	512	0.672459	0.000218	3.971320	0.002159	11.794077	0.000506
200	1024	0.672622	0.000055	3.972939	0.000540	11.794456	0.000127
400	2048	0.672663	0.000014	3.973344	0.000135	11.794551	0.000032
800	4096	0.672674	0.000003	3.973445	0.000034	11.794575	0.000008

Numerical results

Kou model

Table: The rate of ℓ^2-errors obtained by the implicit method with three time levels under the Kou model. The truncated domain is $[-1.5, 1.5]$. N is the number of time steps and M is the number of space steps. q is the rate of convergence defined by (13).

N	M	$\|U(\Delta\tau, \Delta x) - U(\Delta\tau/2, \Delta x/2)\|_{\ell^2}$	q
25	128		
		0.008325398037870	-
50	256		
		0.002108106087901	1.982
100	512		
		0.000526070487931	2.003
200	1024		
		0.000131458391065	2.001
400	2048		
		0.000032860923201	2.000
800	4096		

Conclusions

1. The finite difference method with three time levels to solve the PIDE.
2. Consistency and stability.
3. The second-order convergence in the discrete ℓ^2-norm with a constant ratio $\Delta\tau/\Delta x$.

Thank you for your attention.