Dynamic markov bridges motivated by models of insider trading.

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- Rational Expectations Equilibrium approach: Back (1992), Back and Pedersen (1998), Wu (1999), Cho (2003), Campi and Çetin (2007)

Outline

1. Economic Model

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2. Construction of Dynamic Markov Bridge
   - Problem Formulation
   - Motivation for the Guess
   - Verification of the Guess

3. Equilibrium

Market structure

Consider a market which consists of a single risky asset and a riskless asset with $r = 0$.
Its price at time is denoted by S_t and $S_1 = f(Z_1)$, where Z_t is given by

$$Z_t = Z_0 + \int_0^t \sigma(s) a(V(s), Z_s) dB_s^Z,$$

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2. $V(t) > t$ for every $t \in [0, 1)$, $V(1) = 1$, and $V(t) - t = O(1 - t)$.
3. $a(t, z)$ satisfies a nonlinear PDE:

$$a_t(t, z) + \frac{a^2(t, z)}{2} a_{zz}(t, z) = 0 \quad (1)$$

Market participants

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- Market maker: observes $\mathcal{F}_t^M = \mathcal{F}_t^Y$ where $Y_t = \theta_t + B_t$ is the total order process and sets the price according to

$$H(t, X_t) := S_t = \mathbb{E}[S_1 | \mathcal{F}_t^M]$$

where X_t is strong solution of $dX_t = w(t, X_t)dY_t$, $X_0 = 0$

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1. Market efficiency condition: given θ^*, (H^*, w^*) is a rational pricing rule.
2. Insider optimality condition: given (H^*, w^*), θ^* solves the insider optimization problem:

$$\mathbb{E}^z[W_1^{\theta^*}] = \sup_{\theta \in \mathcal{A}} \mathbb{E}^z[W_1^{\theta}].$$

Characterization of Equilibrium

Lemma *If a triplet* (H^*, w^*, θ^*), *where* (H^*, w^*) *is an admissible pricing rule and* θ^* *is an admissible trading strategy, fulfills the following conditions:*

1. $H_t^*(t,x) + \frac{(w^*(t,x))^2}{2} H_{xx}^*(t,x) = 0.$
2. $w_t^*(t,x) + \frac{(w^*(t,x))^2}{2} w_{xx}^*(t,x) = 0.$
3. $Y_t^* = B_t + \theta_t^*$ *is a standard BM in its own filtration.*
4. $H^*(1, X_1^*) = f(Z_1)$, *where* X^* *is the solution to* $X_t = \int_0^t w(s, X_s) dY_s^*$ *with* $Y^* = B + \theta^*$.
5. $(H^*(t, X_t^*))_{t\in[0,1]}$ *is an* $\mathcal{F}^{Y^*}$*-martingale.*

then it is an equilibrium.

Goal: Given $V(t)$ satisfying assumption 2, and

$$Z_t = Z_0 + \int_0^t \sigma(s) a(V(s), Z_s) dB_s^Z,$$

construct a process X, starting from zero and adapted to $\mathcal{F}_t^{Z,B}$, and measure μ such that:

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C1 $\mathcal{Y} = (\Omega, \mathcal{F}, (\mathcal{F}_t), (X_t, Z_t), (P^{x,z})_{(x,z)\in\mathbb{R}^2})$ is a Markov process, with an initial distribution given by $\delta_0 \otimes \mu$.

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- C2 $X_1 = Z_1$, Q^z-a.s., where Q^z is the law of (X, Z) with $Z_0 = z$ and $X_0 = 0$.
- C3 X with $X_0 = 0$ is a martingale in its own filtration and $[X, X]_t = \int_0^t a^2(s, X_s) ds$.

- Due to Fitzsimmons, Pitman & Yor (1993) (see also Baudoin (2002)), the solution X of

$$dX_t = a(X_t)dB_t + a^2(X_t)\frac{G_x(1-t, X_t, z)}{G(1-t, X_t, z)}dt,$$

where G is the transition density of $d\xi_t = a(\xi_t)d\beta_t$, is a Markov process converging to z as $t \to 1$.

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- If Z_1, independent of B, has a density $G(1, 0, \cdot)$, then

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gives the process we "want": $\mathcal{F}^X$ -martingale with $X_1 = Z_1$.
- Idea: For for $t < 1$, consider $\mu(dz) = \rho(0, x, z)$ and

$$dX_t = a(t, X_t)dB_t + a^2(t, X_t)\frac{\rho_x(t, X_t, Z_t)}{\rho(t, X_t, Z_t)}dt,$$

where $\rho(t, x, z)$ is conditional density of Z_t given X_t.

Restatement of the Problem:

Let

$$A(t,x) := \int_0^x \frac{1}{a(t,y)}dy,$$

and consider $U_t = A(V(t), Z_t)$ and $R_t = A(t, X_t)$. By Itô

$$\begin{aligned} dU_t &= \sigma(t)d\beta_t + \sigma^2(t)b(V(t),U_t)dt \\ dR_t &= dB_t + \left\{\frac{p_x(t,R_t,U_t)}{p(t,R_t,U_t)} + b(t,R_t)\right\}dt, \end{aligned}$$

where $b(t,y) := A_t(t, A^{-1}(t,y)) - \frac{1}{2}a_z(t, A^{-1}(t,y))$.
Then $\rho(t,x,z)$ is conditional density of Z_t with respect to $\mathcal{F}_t^X$ iff $\frac{p(t,x,z)}{a(t,A^{-1}(V(t),z))} := \rho(t, A^{-1}(t,x), A^{-1}(V(t),z))$ is conditional density of U_t with respect to $\mathcal{F}_t^R$.

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- A natural candidate is $p(t,x,z) := \Gamma(t,x;V(t),z)$, where $\Gamma(t,x;s,z)$ is a transition density for $d\zeta_t = d\beta_t + b(t,\zeta_t)dt$.

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Note: Existence of Γ is equivalent to the existence of the fundamental solution of

$$w_u(u, z) = \frac{1}{2}w_{zz}(u, z) - (b(u, z)w(u, z))_z. \qquad (2)$$

Is p the conditional density?

From general filtering theory we have:

$$\widehat{f(U_t)} = \widehat{f(U_0)} + \int_0^t \frac{1}{2}\sigma^2(t)\widehat{\delta_s}ds + \int_0^t \widehat{f(U_s)\kappa_s} - \widehat{f(U_s)}\widehat{\kappa_s}dI_s,$$

where $dI_s = \{dR_s - \widehat{\kappa_s}ds\}$, $\kappa_s := \frac{p_x(s,R_s,U_s)}{p(s,R_s,U_s)} + b(s, R_s)$, $\delta_s = f''(U_s) + 2f'(U_s)b(s, U_s)$ and $(\widehat{H_s})_{s\in[0,1)}$ denotes the optional projection of H given $\mathcal{F}^R$.
Let $g_t(\cdot)$ be the conditional density of U_t given $\mathcal{F}_t^R$. The above suggests that $(g_t(\cdot))_{t\in[0,1)}$ is the weak solution to the SPDE

$$g_t(z) = \Gamma(0,0;c,z) + \int_0^t \sigma^2(s)\left\{-(b(s,z)g_s(z))_z + \tfrac{1}{2}(g_s(z))_{zz}\right\}ds$$
$$+ \int_0^t g_s(z)\left(\frac{p_x(s,R_s,z)}{p(s,R_s,z)} - \int_{\mathbb{R}} g_s(z)\frac{p_x(s,R_s,z)}{p(s,R_s,z)}dz\right)dI_s.$$

Is R_t well-defined for all $t \in [0,1]$?

We have:

$$\begin{aligned} dU_t &= \sigma(t)d\beta_t + \sigma^2(t)b(t,U_t)dt \\ dR_t &= dB_t + \left\{ \frac{p_x(t,R_t,U_t)}{p(t,R_t,U_t)} + b(t,R_t) \right\} dt, \end{aligned}$$

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Assumption 4 b, b_y and b_t are uniformly bounded on $[0, 1] \times \mathbb{R}$, and b_y is Hölder continuous uniformly in t.
Proposition: *Suppose that Assumptions 2 and 4 hold. Then* $Q^z(\lim_{t\to 1} R_t = U_1) = 1$

So, is p conditional density?

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$$\mathcal{A}_0\phi(t,x) = \frac{\partial \phi}{\partial t}(t,x) + \frac{1}{2}\sigma^2(t)\frac{\partial^2 \phi}{\partial x^2}(t,x) + \sigma^2(t)b(t,x)\frac{\partial \phi}{\partial x}(t,x),$$

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the corresponding martingale problem is well-posed $\Rightarrow$ can modify arguments of Kurtz-Ocone (1988) and obtain that

$$\pi_t f = \pi_0 f + \int_0^t \pi_s(\mathcal{A}_0 f)ds + \int_0^t [\pi_s(\kappa_s f) - \pi_s\kappa_s\pi_s f]\, d\left\{R_t - \int_0^t \pi_s\kappa_s ds\right\}$$

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has unique solution. $\Rightarrow$ p is conditional density of U
$\Rightarrow$ $\rho(t,x,x) = \frac{p(t,A(x,t),A(z,V(t)))}{a(t,z)} := G(t,x,V(t),z)$ is conditional density of Z, where G is a transition density of $d\eta_t = a(t,\eta_t)d\beta_t$.

Is X a local martingale?

Since

$$dX_t = a(t, X_t)dB_t + a^2(t, X_t)\frac{\rho_x(t, X_t, Z_t)}{\rho(t, X_t, Z_t)}dt,$$

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it follows that

$$dX_t = a(t, X_t)dB_t^X + a^2(t, X_t)\mathbb{E}\left[\frac{\rho_x(t, X_t, Z_t)}{\rho(t, X_t, Z_t)}\middle|\mathcal{F}_t^X\right]dt,$$

where B^X is an $\mathcal{F}^X-$Brownian motion.

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where B^X is an $\mathcal{F}^X-$Brownian motion. However, since $\rho(t, X_t, \cdot) = G(t, X_t; V(t), z)$ is the conditional density of Z_t,

$$\mathbb{E}\left[\frac{\rho_x(t, X_t, Z_t)}{\rho(t, X_t, Z_t)}\middle|\mathcal{F}_t^X\right] = \int_{\mathbb{R}} G_x(t, X_t; V(t), z)dz = 0$$

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$\Rightarrow$ X is local martingale

Main Result

Theorem Suppose that Assumptions 2 and 4 hold, and $\mu(dz) = G(0,0;c,z)dz$. Let for $t < 1$

$$dX_t = a(t,X_t)dB_t + a^2(t,X_t)\frac{\rho_x(t,X_t,Z_t)}{\rho(t,X_t,Z_t)}dt,$$

where $\rho(t,x,z) := G(t,x;V(t),z)$, on every interval $[0,T]$ with $T < 1$, there exists a unique strong solution to the above SDE with the initial condition $X_0 = 0$. Moreover, the conditions C1-C3 are satisfied.

Theorem *Under Assumptions 1-4, there exists an equilibrium* (H^*, w^*, θ^*), *where*

(i) $H^*(t,x) = \int_{\mathbb{R}} f(y)G(t,x;1,y)\,dy$, *and* $w^*(t,x) = a(t,x)$ *for all* $(t,x) \in [0,1] \times \mathbb{R}$;

(ii) $\theta^*_t = \int_0^t \alpha^*_s ds$ *where* $\alpha^*_s = a(s,X_s)\frac{\rho_x(s,X_s,Z_s)}{\rho(s,X_s,Z_s)}$ *and the process* X *is the unique strong solution under* $\mathcal{F}^{B,Z}$ *of the following SDE:*

$$dX_t = a(t,X_t)dB_t + a^2(t,X_t)\frac{\rho_x(t,X_t,Z_t)}{\rho(t,X_t,Z_t)}dt, \quad X_0 = 0.$$