

Fast Pricing and Calculation of Sensitivities of OTM European Options Under Lévy Processes

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Outline

- 1 Motivation and Main Results
- 2 Pricing of European options in Lévy Models
- 3 Integration-Along-Cut (IAC) Method
- 4 Sensitivities
- 5 Realization of IAC in KoBoL Model
- 6 More Numerical Examples

Motivation

- FFT method:
 - ▶ first applied to pricing European options by Carr and Madan [1999]
 - ▶ sizable computational error for deep OTM options
- Integration-Along-Cut (IAC) method:
 - ▶ Levendorskii and Zherder [2001], Boyarchenko and Levendorskii [2002]
- Later, deficiencies of FFT techniques were analyzed by
 - ▶ Lord, Fang, Bervoets and Oosterlee [2007]
 - ▶ Lord and Kahl [2007]
 - ▶ N. Boyarchenko and Levendorskii [2007]
 - ▶ M. Boyarchenko and Levendorskii [2008, 2009] (refined and enhanced (enh-ref) FFT)
 - ▶ Carr and Madan [2009] (saddlepoint method)

Main Results

- 1 Accurate and fast numerical realizations for KoBoL (a.k.a. CGMY) and VG model
- 2 A modification, which makes IAC method applicable to European options far from expiry in VG model
- 3 An efficient procedure for calculation of option prices at many strikes
- 4 Calculation of sensitivities using IAC method

Study of Relative Efficiency: An Example

Table: Prices of European call options in the KoBoL model: IAC vs. FFT, the approximate saddle point method saddlepoint (SP) and enh-ref iFFT (Enh-ref).

Strike	Option price				Relative difference between IAC and		
	IAC	FFT	SP	Enh-ref	FFT	SP	Enh-ref
130	3.6628	3.6601	3.6387	3.6608	-0.000729	-0.006572	-0.000540
140	2.3350	2.3324	2.3225	2.3330	-0.001132	-0.005372	-0.000888
150	1.4867	1.4840	1.4879	1.4848	-0.001831	0.000792	-0.001289
160	0.9509	0.9482	0.9537	0.9493	-0.002886	0.002898	-0.001732
170	0.6134	0.6012	0.6159	0.6120	-0.019886	0.004078	-0.002207
180	0.4000	0.3278	0.4018	0.3989	-0.180406	0.004615	-0.002707
190	0.2639	0.0129	0.2652	0.2631	-0.951127	0.004744	-0.003227
200	0.1764	-0.4588	0.1772	0.1757	-3.600770	0.004482	-0.003763

European call option parameters: $r = 0.03$, $T = 0.5$, $S = 100$.

KoBoL parameters: $\nu = 0.5$, $c_+ = c_- = 2$, $\lambda_+ = 5$, $\lambda_- = -10$, $\mu \approx 0.3926$.

Enh-ref iFFT algorithm parameters (CPU time: 0.06 seconds): $M = 2^{13}$, $M_2 = 2$, $M_3 = 2$, $\Delta = 0.00125$.

Study of Relative Efficiency: An Example (cont.)

- The second column is obtained by using Integration-Along-Cut method for a given relative error tolerance $5 \cdot 10^{-5}$.
- The third and fourth column are taken from the tables in [P. Carr and D.B. Madan](#), *Saddlepoint Methods for Option Pricing*, Journal of Computational Finance, Vol. 13, No. 1 (Fall 2009).
- Refined and enhanced iFFT is a flexible version of iFFT developed in [M. Boyarchenko and S. Levendorskii](#), *Prices and sensitivities of barrier and first-touch digital options in Lévy-driven models*, International Journal of Theoretical and Applied Finance, Vol. 12, No. 8, 2009, pp. 1125–1170

N.B.: For a very long and fine grid, enh-ref iFFT method produces results with relative differences less than $5 \cdot 10^{-5}$, but this requires much more CPU time than IAC

Speed: An Example in VG Model

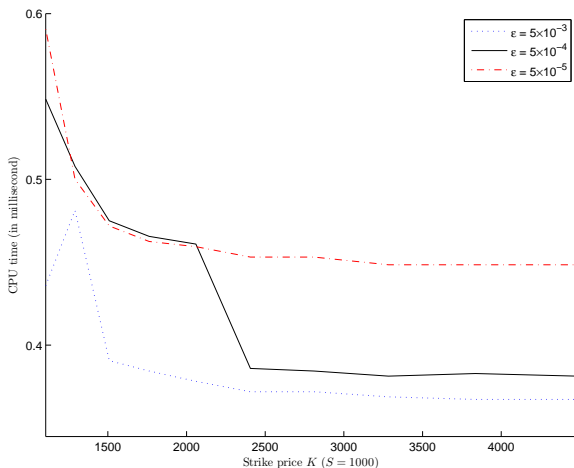


Figure: CPU time (in millisecond) of IAC method, for the European call option in VG model: dependence on the relative error tolerance ϵ . *European call option parameters:* $r = 0.01$, $T = 0.5$, $S = 1000$. *VG model parameters:* $c_+ = c_- = 0.15$, $\lambda_+ = 9$, $\lambda_- = -8$, $\mu \approx 0.0058$.

Speed: An Example in KoBoL Model

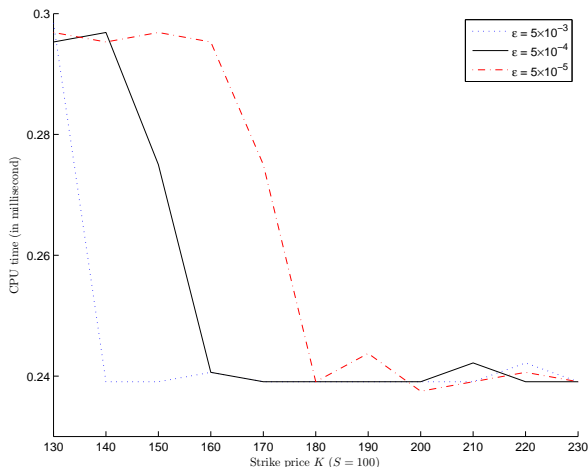


Figure: CPU time (in millisecond) of IAC method, for the European call option in KoBoL model: dependence on the relative error tolerance ϵ . *European call option parameters:* $r = 0.03$, $T = 0.5$, $S = 100$. *KoBoL parameters:* $\nu = 0.5$, $c_+ = c_- = 2$, $\lambda_+ = 5$, $\lambda_- = -10$, $\mu \approx 0.3926$.

Strong Features of Our Approach

Application

Variance Gamma (VG) model and strongly Regular Lévy Process of Exponential type (sRLPE), which includes Normal Inverse Gaussian (NIG) model, and KoBoL (a.k.a CGMY) model in the finite variation case

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Accuracy & Efficiency

- For one strike, hundreds times faster than FFT
- For multiple options, IAC method together with the quadratic interpolation, is still faster and more accurate than FFT based approach
- For sensitivities, a relative advantage of IAC is even greater

Strong Features of Our Approach (cont.)

Efficient Error Control

Real-valued integrand, and

- For VG, a simple rule for truncation; and asymptotic expansion may be used to calculate the truncated part
- For KoBoL, the calculations are reduced to summation of oscillating series; even easier

Strong Features of Our Approach (cont.)

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Simplicity

- Straightforward reduction to cuts
- Simpson's and enhanced Simpson's (a.k.a product integration) rule

Pricing of European Options in Lévy Models

Market structure

- Bond — yielding riskless rate of return r
- Stock — $S_t = e^{X_t}$, where X_t is a Lévy process under an EMM \mathbb{Q} ; the characteristic exponent $\psi(\xi)$ of X_t admits the analytic continuation into the complex plane with two cuts $i(-\infty, \lambda_-]$ and $i[\lambda_+, +\infty)$

Notation: $V(t, x) := V(G; T; t, x)$ — the price of the European option with maturity T and payoff $G(X_T)$

Lévy processes: General Definitions

ψ — characteristic exponent of $X = (X_t)$:

$$E \left[e^{i\xi X_t} \right] = e^{-t\psi(\xi)}$$

Explicit formula (Lévy-Khintchine formula, in 1D) is

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \int_{\mathbb{R}\setminus 0} (1 - e^{iy\xi} + iy\xi \mathbb{1}_{|y|<1}(y))F(dy),$$

where $F(dy)$, the Lévy density, satisfies

$$\int_{\mathbb{R}\setminus 0} \min\{|y|^2, 1\}F(dy) < \infty.$$

Examples

a) *KoBoL model* (a.k.a. CGMY model and extended Koponen's family):

$$\psi(\xi) = -i\mu\xi + \Gamma(-\nu) \cdot [c_+((- \lambda_-)^\nu - (-\lambda_- - i\xi)^\nu) + c_- (\lambda_+^\nu - (\lambda_+ + i\xi)^\nu)],$$

where $\nu \neq 1$, $c_\pm > 0$, $\lambda_- < 0 < \lambda_+$.

b) *Normal Tempered Stable Lévy processes* (NTS):

$$\psi(\xi) = -i\mu\xi + \delta \cdot [(\alpha^2 - (\beta + i\xi)^2)^{\nu/2} - (\alpha^2 - \beta^2)^{\nu/2}],$$

where $\alpha > |\beta| > 0$, $\delta > 0$ and $\mu \in \mathbb{R}$. With $\nu = 1$, NTS is NIG.

c) *Variance Gamma process* (VG):

$$\psi(\xi) = -i\mu\xi + c_+ [\ln(-\lambda_- - i\xi) - \ln(-\lambda_-)] + c_- [\ln(\lambda_+ + i\xi) - \ln(\lambda_+)],$$

where $c_\pm > 0$, $\lambda_- < 0 < \lambda_+$, $\mu \in \mathbb{R}$.

Generalized Black-Scholes Formula

Boyarchenko and Levendorskii [1998], Carr and Madan [1998]

Assume that for $\omega \in (\lambda_-, \lambda_+)$, function $G_\omega(x) := e^{\omega x} G(x) \in L_1(\mathbb{R})$. Then

$$V(t, x) = \mathbb{E} \left[e^{-r(T-t)} G(X_T) \mid X_t = x \right]$$

$$G(x) = (2\pi)^{-1} \int_{\text{Im } \xi = \omega} e^{ix\xi} \hat{G}(\xi) d\xi,$$

$$V(t, x) = (2\pi)^{-1} \int_{\text{Im } \xi = \omega} e^{ix\xi - \tau(r + \psi(\xi))} \hat{G}(\xi) d\xi, \quad (1)$$

where $\tau = T - t > 0$.

Equivalently,

$$V(t, x) = (2\pi)^{-1} \int_{\text{Im } \xi = \omega} e^{i(x + \mu\tau)\xi - \tau(r + \psi^0(\xi))} \hat{G}(\xi) d\xi. \quad (2)$$

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Generalized Black-Scholes Formula (cont.)

Let $x = \ln(S_t/K)$.

- **European call option:** $G(X_T) = (e^{X_T} - K)_+$

$$V(t, x) = -\frac{Ke^{\tau r}}{2\pi} \int_{\text{Im } \xi = \omega} \frac{e^{i(x + \mu\tau)\xi - \tau\psi^0(\xi)}}{(\xi + i)\xi} d\xi. \quad (3)$$

for any $\omega \in (\lambda_-, -1)$.

- **European put option:** $G(X_T) = (K - e^{X_T})_+$, we have the same formula as above, but with $\omega \in (0, \lambda_+)$.

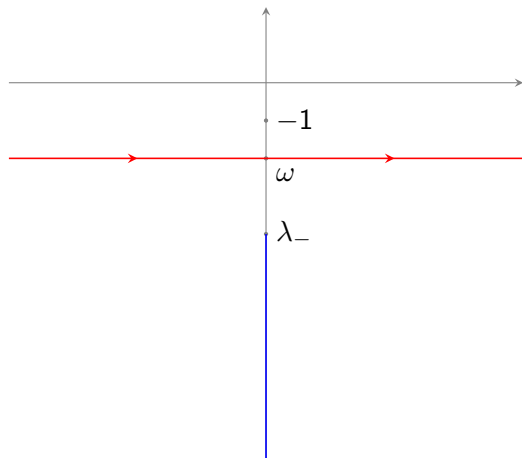
Definition

Let $\lambda_- < 0 < \lambda_+$ and $\nu \in (0, 2]$. We call X a *strongly regular Lévy process of exponential type* (λ_-, λ_+) and *order* ν if the following conditions hold:

- (i) the characteristic exponent ψ admits the analytic continuation into the complex plane with two cuts $i(-\infty, \lambda_-]$ and $i[\lambda_+, +\infty)$;
- (ii) for $z \leq \lambda_-$ and $z \geq \lambda_+$, the limits $\psi(iz \pm 0)$ exist;
- (iii) there exists $\mu \in \mathbb{R}$ such that the function $\psi^0(\xi) := \psi(\xi) + i\mu\xi$ is asymptotically positively homogeneous of order ν as $\xi \rightarrow \infty$ in the complex plane with these cuts.

In VG model, the logarithmic asymptotics at infinity: sRLPE of order $0+$.

Idea of IAC Method



Example: European OTM Call Option ($\lambda_- < \omega < -1$, $x + \mu\tau < 0$)

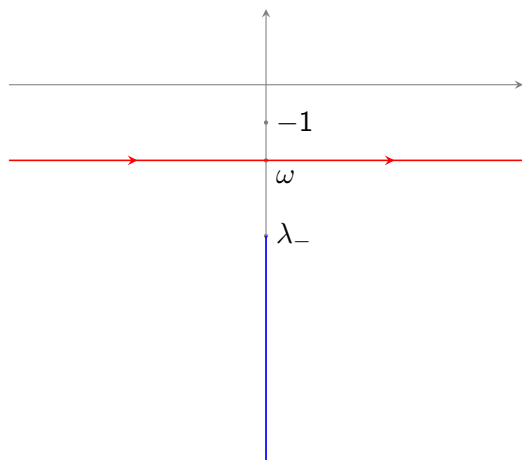
Assumption: X is an sRLPE of order $\nu \in (0, 1]$

Integral:

$$\int_{\text{Im } \xi = \omega} \phi(\xi) d\xi$$

$$\phi(\xi) = e^{i(x+\mu\tau)\xi - \tau\psi^0(\xi)} \frac{1}{(\xi + i)\xi}$$

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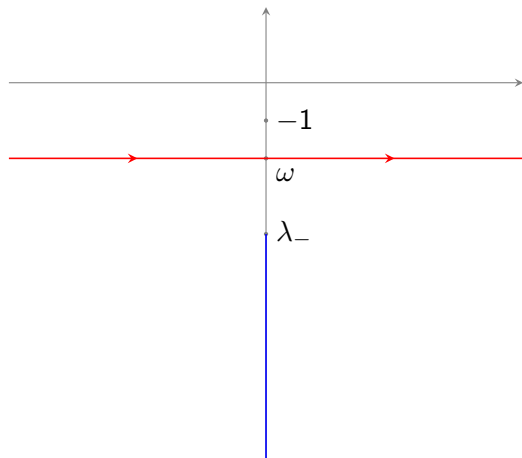
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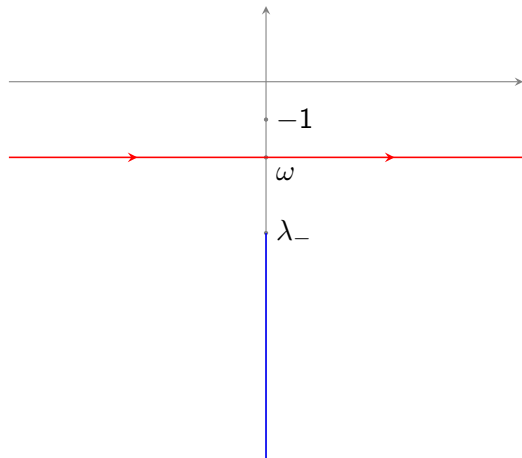
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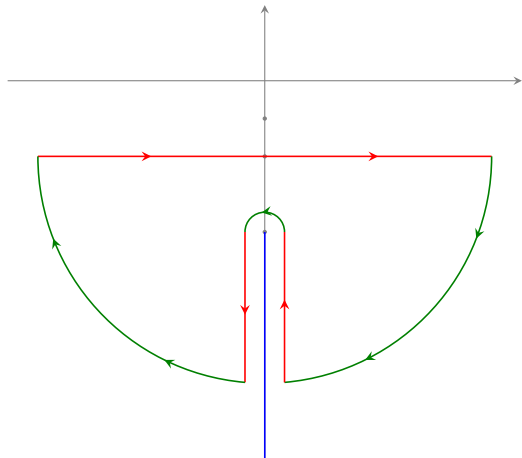
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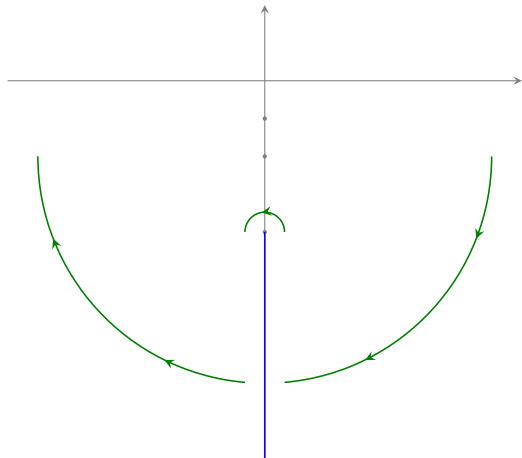
Idea of IAC Method



By Cauchy theorem,

$$\oint \phi(\xi) d\xi = \int_{\mathcal{L}_\omega} \phi(\xi) d\xi + \int_{\mathcal{L}_R} \phi(\xi) d\xi + \int_{\mathcal{L}_e} \phi(\xi) d\xi + \int_{\mathcal{L}_{\lambda_-}} \phi(\xi) d\xi = 0$$

Idea of IAC Method



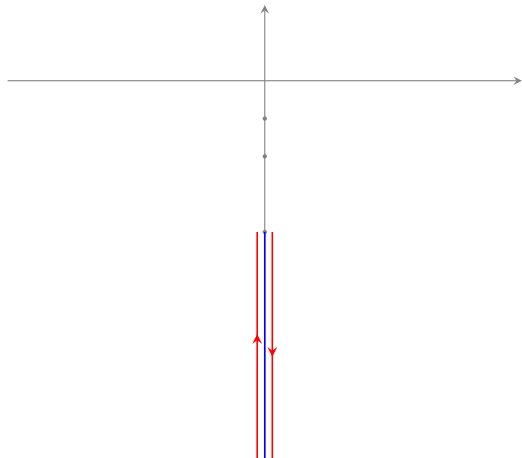
Let $\text{Im } \xi \leq \omega$. As $R := |\xi| \rightarrow +\infty$,

$$\int_{\mathcal{L}_R} \phi(\xi) d\xi \rightarrow 0$$

As $\epsilon \rightarrow 0$,

$$\int_{\mathcal{L}_\epsilon} \phi(\xi) d\xi \rightarrow 0$$

Idea of IAC Method



$$\begin{aligned} & \int_{\text{Im } \xi = \omega} \phi(\xi) d\xi \\ &= \int_{-i\infty-0}^{i\lambda_--0} \phi(\xi) d\xi \\ & \quad + \int_{i\lambda_++0}^{-i\infty+0} \phi(\xi) d\xi \end{aligned}$$

IAC Formula for European OTM Call and Put Option

- OTM Call ($x' := -x - \mu\tau > 0$)

$$V_{\text{call}}(t, x) = \frac{Ke^{-r\tau}}{\pi} \int_{-\infty}^{\lambda_-} e^{zx'} \frac{\Omega(\tau, z)}{z(z+1)} dz, \quad (4)$$

- OTM Put ($x' := x + \mu\tau > 0$)

$$V_{\text{put}}(t, x) = -\frac{Ke^{-r\tau}}{\pi} \int_{\lambda_+}^{+\infty} e^{-zx'} \frac{\Omega(\tau, z)}{z(z+1)} dz, \quad (5)$$

where

$$\Omega(\tau, z) = (i/2)(e^{-\tau\psi^0(iz-0)} - e^{-\tau\psi^0(iz+0)}). \quad (6)$$

$\Omega(\tau, z)$ in The Case of KoBoL

- For $z < \lambda_-$:

$$\begin{aligned}\Omega(\tau, z) &= -e^{-\tau c \Gamma(-\nu)} [(-\lambda_-)^\nu - (\lambda_- - z)^\nu \cos(\nu\pi) + \lambda_+^\nu - (\lambda_+ - z)^\nu] \\ &\quad \times \sin(\tau c \Gamma(-\nu)) (\lambda_- - z)^\nu \sin(\nu\pi); \end{aligned}$$

- For $z > \lambda_+$:

$$\begin{aligned}\Omega(\tau, z) &= e^{-\tau c \Gamma(-\nu)} [(-\lambda_-)^\nu - (-\lambda_- + z)^\nu + \lambda_+^\nu - (z - \lambda_+)^\nu \cos(\nu\pi)] \\ &\quad \times \sin(\tau c \Gamma(-\nu)) (z - \lambda_+)^\nu \sin(\nu\pi); \end{aligned}$$

Sensitivities

Denote by $S = Ke^x$, the spot price of the underlying. For a European call option,

$$\Delta = \partial V / \partial S = \frac{e^{-x}}{2\pi} \int_{\text{Im } \xi = \omega} \frac{e^{i(x+\mu\tau)\xi - \tau(r+\psi^0(\xi))}}{i\xi - 1} d\xi;$$

$$\Gamma = \partial^2 V / \partial S^2 = \frac{e^{-2x}}{2\pi K} \int_{\text{Im } \xi = \omega} e^{i(x+\mu\tau)\xi - \tau(r+\psi^0(\xi))} d\xi,$$

$$\Theta = -\partial V / \partial \tau = -\frac{K}{2\pi} \int_{\text{Im } \xi = \omega} \frac{e^{i(x+\mu\tau)\xi - \tau(r+\psi^0(\xi))}}{(\xi + i)\xi} (r - i\mu\xi + \psi^0(\xi)) d\xi.$$

Sensitivities (cont.)

If X is an sRLPE of order $\nu \in (0, 1]$, then

$$\begin{aligned}\Delta &= \frac{e^{-x-\tau r}}{\pi} \int_{-\infty}^{\lambda_-} e^{x'z} \frac{\Omega(\tau, z)}{-z-1} dz \\ \Gamma &= \frac{e^{-2x-\tau r}}{\pi K} \int_{-\infty}^{\lambda_-} e^{x'z} \Omega(\tau, z) dz, \\ \Theta &= \frac{Ke^{-\tau r}}{\pi} \int_{-\infty}^{\lambda_-} e^{x'z} \frac{\Omega_1(\tau, z)}{z(z+1)} dz\end{aligned}$$

where

$$\begin{aligned}\Omega_1(\tau, z) &= \frac{i}{2} \left(e^{-\tau\psi^0(iz-0)} [r + \mu z + \psi^0(iz-0)] \right. \\ &\quad \left. - e^{-\tau\psi^0(iz+0)} [r + \mu z + \psi^0(iz+0)] \right)\end{aligned}$$

Realization of IAC in KoBoL Model

Change the variable $y = (\lambda_- - z)^\nu$ in (4):

$$V(t, x) = R_\nu \int_0^\infty e^{-x'y^{1/\nu}} \cdot F(y) \cdot \sin(\delta_+ \sin(\nu\pi)y) dy$$

where $x' = -x - \mu\tau > 0$, $\delta_\pm = -\tau c_\pm \Gamma(-\nu)$,

$$R_\nu = Ke^{-r\tau + x'\lambda_- + \delta_+ (-\lambda_-)^\nu + \delta_- \lambda_+^\nu} / (\nu\pi),$$

$$F(y) = \frac{e^{-\delta_+ \cos(\nu\pi)y - \delta_- (\lambda_+ - \lambda_- + y^{1/\nu})^\nu}}{(\lambda_- - y^{1/\nu})(\lambda_- + 1 - y^{1/\nu})} \cdot y^{1/\nu - 1}$$

Realization of IAC in KoBoL Model (cont.)

Set $na = n\pi / (\delta_+ \sin(\nu\pi))$, $n = 0, 1, \dots$, and change the variable $y = z + na$:

$$V(t, x) = R_\nu \sum_{n=0}^{\infty} (-1)^n \int_0^a F_n(x, z) dz,$$

where

$$F_n(x, z) = e^{-x'(z+na)^{1/\nu}} \cdot f_n(z)$$

$$f_n(z) = \frac{\sin(\pi z/a) \cdot e^{-\delta_+ \cos(\nu\pi)(z+na) - \delta_- (\lambda_+ - \lambda_- + (z+na)^{1/\nu})^\nu}}{(\lambda_- - (z+na)^{1/\nu}) \cdot (\lambda_- + 1 - (z+na)^{1/\nu}) \cdot (z+na)^{1-1/\nu}}$$

Realization of IAC in KoBoL Model (cont.)

Lemma

Let $z \in (0, a)$. Then

- a) if $\nu \in (0, 0.5]$, then the sequence $\{F_n(z)\}$ is monotonically decreasing;
- b) if $\nu \in (0.5, 1)$, then there exists N such that for $n \geq N$, the sequence $\{F_n(z)\}$ is monotonically decreasing.

Example: Fitting The Smile

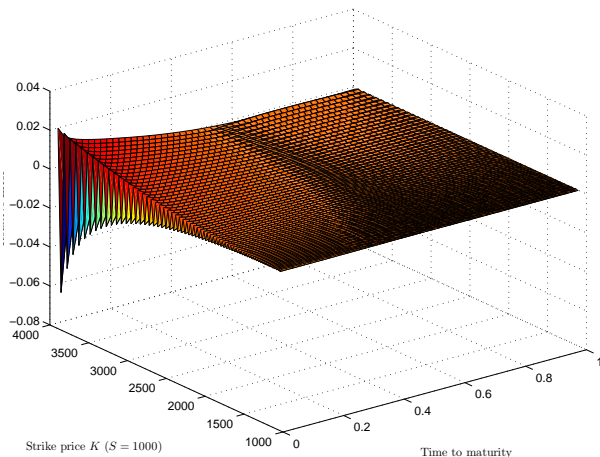


Figure: Relative error of implied volatility surface obtained by enh-ref iFFT method. VG parameters: $c_+ = c_- = 0.15$, $\lambda_+ = 9$, $\lambda_- = -8$, $\mu \approx 0.0058$. European call option parameters: $r = 0.01$, $S = 1000$. IAC (multiple options) algorithm parameters: $\Delta_1 = 0.0625$, $\Delta_2 = 0.5$, $\epsilon = 5 \times 10^{-5}$, $A(x', \epsilon) = A(-0.1, 5 \times 10^{-5})$, $\Delta_x = 0.02$. Enh-ref iFFT algorithm parameters: $\Delta = 0.01$, $M = 2^{10}$, $M_2 = 3$, $M_3 = 3$. Relative error: $(V_{\text{enh-ref}} - V_{\text{IAC}})/V_{\text{IAC}}$.

Example: Fitting The Smile (cont.)

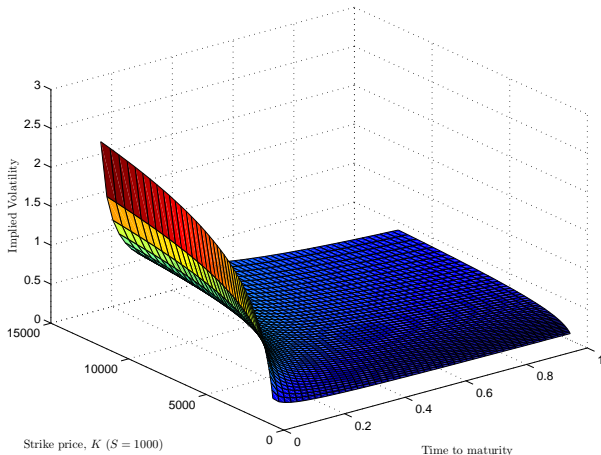


Figure: Implied volatility surface. VG parameters: $c_+ = c_- = 0.15$, $\lambda_+ = 9$, $\lambda_- = -8$, $\mu \approx 0.0058$. European call option parameters: $r = 0.01$, $S = 1000$. IAC (multiple options) algorithm parameters: $\Delta_1 = 0.0625$, $\Delta_2 = 0.5$, $\epsilon = 5 \times 10^{-5}$, $A(x', \epsilon) = A(-0.1, 5 \times 10^{-5})$, $\Delta_x = 0.02$.

Possible Efficiency Improvement

- C++ implementation with the x' -dependent truncation parameter
- Non-uniformly spaced grid

More Results

S.Z. Levendorskiĭ and J. Xie, *Fast pricing and calculation of sensitivities of out-of-the-money European options under Lévy processes* (working paper, April 14, 2010). Available at SSRN:

<http://ssrn.com/abstract=1589809>

PC Characterization

The calculations presented were performed in MATLAB©7.4.0 (R2007a), on a PC with characteristics Intel®Core™2 Duo CPU (3.16GHz, 6MB L2 Cache, 1333MHz FSB), under the Genuine Windows®XP Professional operating system.