Positive Stochastic Volatility Simulation

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Introduction: The Heston Model

$$\mathrm{d}S_t = \mu S_t \,\mathrm{d}t + \sqrt{V_t}\, S_t \left(\rho \mathrm{d}W_t^1 + \sqrt{1-\rho^2}\,\mathrm{d}W_t^2\right),$$
$$\mathrm{d}V_t = \kappa(\theta - V_t)\,\mathrm{d}t + \varepsilon\sqrt{V_t}\,\mathrm{d}W_t^1,$$

where

- W_t^1 and W_t^2 independent scalar Wiener processes
- μ, κ, θ and ε are positive constants
- $\rho \in (-1, 1)$
- S price process of underlying variable (e.g. stock index, exchange rate)
- V variance process.

Heston (1993), Cox, Ingersoll & Ross (1985), Feller (1951)

Properties of V

- $V_t \geq 0$ (assuming $V_0 \geq 0$).
- Let
$$\nu := 4\kappa\theta/\varepsilon^2,$$
  Then
  - If $\nu \geq 2$, then V is strictly positive.
  - If $\nu < 2$, then the zero boundary is attainable and instantaneously reflecting.
- Attainability of zero boundary and reflection property are major obstacle in computational treatment.
- $\nu < 2$ is relevant for foreign exchange and long-dated interest-rate markets (Andersen 2008).

Here: focus on attainable zero boundary case, in particular $\nu < 1$.

Modifications of Euler-Maruyama Scheme

Extend vector fields to negative domain. e.g.

- Partial truncation (Delbaen and Deelstra 1998)

$$\hat{V}_{t_{n+1}} = \hat{V}_{t_n} + h\kappa(\theta - \hat{V}_{t_n}) + \varepsilon \Delta W^1_{t_n} \sqrt{\hat{V}^+_{t_n}} \,.$$

- Reflection (Bossy and Diop 2007)

$$\hat{V}_{t_{n+1}} = |\hat{V}_{t_n}| + h\kappa(\theta - |\hat{V}_{t_n}|) + \varepsilon \Delta W^1_{t_n} \sqrt{|\hat{V}_{t_n}|} \,.$$

- Full truncation (Lord, Koekoek & Van Dijk 2006)

$$\hat{V}_{t_{n+1}} = \hat{V}_{t_n} + h\kappa(\theta - \hat{V}^+_{t_n}) + \varepsilon \Delta W^1_{t_n} \sqrt{\hat{V}^+_{t_n}} \,.$$

Full truncation works well, but properties (e.g. error) are difficult to derive.

Transition Probability for V

$$\mathbb{P}\big(V_t < x \;\big|\; V_0\big) = F_{\chi^2_\nu(\lambda)}\big(x \cdot \eta(t) / \exp(-\kappa t)\big),$$

where

- $F_{\chi^2_\nu(\lambda)}$ non-central chi-squared distribution function with ν degrees of freedom and non-centrality parameter λ

$$F_{\chi^2_\nu(\lambda)}(z) = \frac{\mathrm{e}^{-\lambda/2}}{2^{\nu/2}} \sum_{j=0}^{\infty} \frac{(\lambda/2)^j}{j! 2^j \Gamma(\nu/2 + j)} \int_0^z \xi^{\nu/2+j-1} \mathrm{e}^{-\xi/2} \, \mathrm{d}\xi,$$

- $\nu := 4\kappa\theta/\varepsilon^2$,
- $\eta(t) := \frac{4\kappa \exp(-\kappa t)}{\varepsilon^2 \big(1 - \exp(-\kappa t)\big)}$,
- $\lambda = V_0 \eta(t)$.

Properties of Chi-Square Distribution

- *Dealing with Non-Centrality*

$$\chi^2_\nu(\lambda) = \chi^2_{\nu+2N},$$

  where N is Poisson distributed with parameter $\lambda/2$. (Johnson 1959, Glasserman 2003)
- *Divisibility of Chi-Squared Distribution:* Assume
  - $Y_1, Y_2, \ldots,, Y_{2N}, Z$ independent,
  - Y_i standard Normally distributed, $i = 1, \ldots, 2N$,
  - Z χ^2_ν-distributed.

  Then

$$\sum_{i=1}^{2N} Y_i^2 + Z \sim \chi^2_{\nu+2N}.$$

Questions:

- How to simulate a χ^2_ν random variable for non-integer $\nu < 1$?
- Is there a representation for a χ^2_ν variable with non-integer $\nu < 1$ similar to the integer degrees of freedom case?

Generalized Gaussian Distribution

Definition: A generalized $\mathrm{N}(0, 1, q)$ random variable, for $q \geq 1$, has density

$$f_{\mathrm{N}(0,1,q)}(x) := \frac{q}{2^{1/q+1}\Gamma(1/q)} \cdot \exp\left(-\tfrac{1}{2}|x|^q\right),$$

where $x \in \mathbb{R}$ and $\Gamma(\cdot)$ is the standard gamma function.

Note that for $q = 2$, we recover the Normal distribution.

(Gupta & Song 1997, Song & Gupta 1997, Sinz, Gerwinn & Bethge 2009, Sinz & Bethge 2008)

Representation of Chi-Square by Generalized Gaussian

Theorem:
Suppose $X_i \sim \mathrm{N}(0, 1, 2q)$ are independent identically distributed random variables for $i = 1, \ldots, p$, where $q \geq 1$ and $p \in \mathbb{N}$. Then we have

$$\sum_{i=1}^{p} \left| X_i \right|^{2q} \sim \chi^2_{p/q}.$$

Proof: Calculate density.

Generalized Marsaglia Polar Method

Theorem:
Suppose for some $q \in \mathbb{N}$ that $U_1, \ldots, U_q$ are independent identically distributed uniform random variables over $[-1, 1]$. Condition this sample set to satisfy the requirement $\|U\|_q < 1$, where $\|U\|_q$ is the q-norm of $U = (U_1, \ldots, U_q)$. Then the q random variables generated by $U \cdot (-2 \log \|U\|_q^q)^{1/q} / \|U\|_q$ are independent $N(0, 1, q)$ distributed random variables.

Proof: Calculate density.

Remark: $q = 2$: Marsaglia's Polar Method for Normal distribution.

Summary: Algorithm

Assume $\nu = \frac{p}{q}$ with p and q natural numbers (no loss of generality). To produce an exact $\chi^2_{p/q}(\lambda)$ sample:

1. Generate $2q$ independent uniform random variables over $[-1, 1]$: $U = (U_1, \ldots, U_{2q})$.
2. If $\|U\|_{2q} < 1$ continue, otherwise repeat Step 1.
3. Compute $Z = U \cdot (-2 \log \|U\|_{2q}^{2q})^{1/2q} / \|U\|_{2q}$. This gives $2q$ independent $\mathrm{N}(0, 1, 2q)$ random variables $Z = (Z_1, \ldots, Z_{2q})$.
4. Compute $Z_1^{2q} + \cdots + Z_p^{2q} \sim \chi^2_{p/q}(\lambda)$.

Probability of Acceptance

Probability of acceptance in each attempt:

$$p_M = \left(\frac{\Gamma\left(1/2q\right)}{2q} \right)^{2q}$$

- $q = 1$: probability of acceptance is 0.7854 (sample from Gaussian distribution)
- As $q \to \infty$, we have $p_M \to \mathrm{e}^{-\gamma} \approx 0.5615$, where γ is the Euler-Mascheroni constant.
- In each accepted attempt, $2q$ independent generalized Gaussian variables are generated, of which $p < q$ are used to generate one $\chi^2_{p/q}$ variable.
- Expected number of attempts to generate $2q/p$ independent $\chi^2_{p/q}$ variables is

$$1/p_M \in [1.2732,\, 1.7809].$$

Comparison with Acceptance-Rejection Method

Ahrens & Dieter: acceptance-rejection method with mixture of prior densities

- $(p/2q)x^{p/2q-1}$ on $[0,1]$ with weight $\mathrm{e}/(\mathrm{e}+p/2q)$,
- $\exp(1-x)$ on $[1,\infty)$ with weight $(p/2q)(\mathrm{e}+p/2q)$.

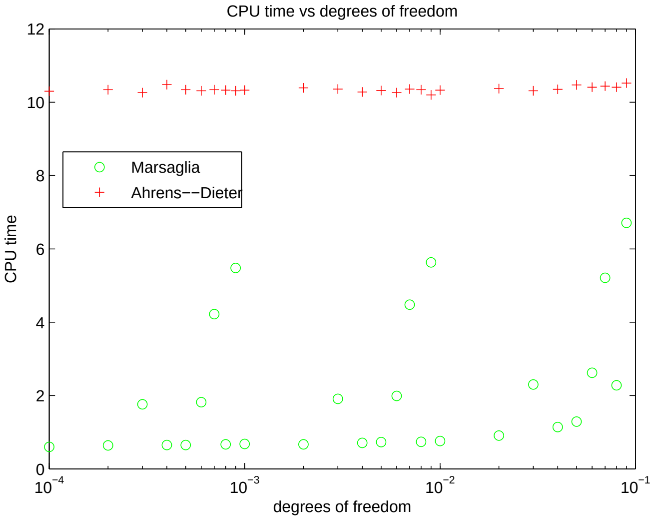
- Expected number of steps to generate $2q/p$ independent $\chi^2_{p/q}$ variables is

$$(2q/p)\cdot\frac{p/2q+\mathrm{e}}{p/2q\Gamma(p/2q)\mathrm{e}}.$$

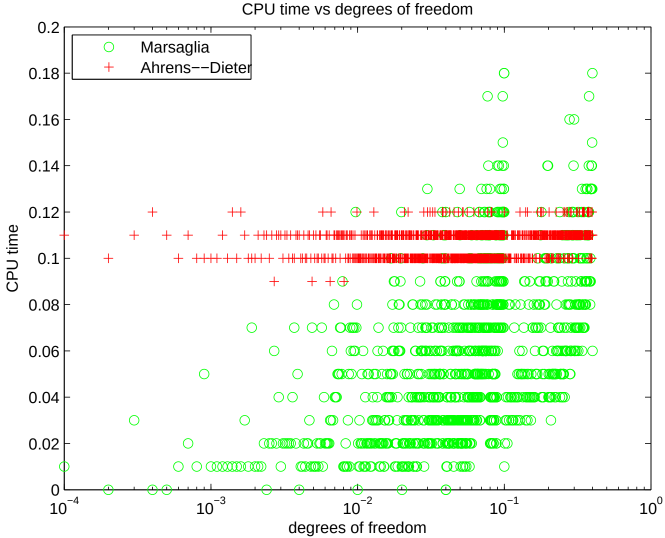
  This is
  - larger than expected number of steps in generalized Marsaglia method for all $p/q<1$.
  - unbounded.

- Computational effort (CPU time): generalized Marsaglia method compares very favourably with acceptance-rejection method (see overleaf).

CPU Time vs Degrees of Freedom – 1 Digit



CPU Time vs Degrees of Freedom – 3 Significant Digits



Andersen's Distribution Approximation

- If $\hat{V}_{t_n}$ is large:
  $$\hat{V}_{t_{n+1}} = (a + bZ)^2 \,,$$
  where $Z \sim N(0,1)$.
- If $\hat{V}_{t_n}$ is small: replace true density with mixture of Dirac delta function and exponential density
  $$p\delta(0) + (1-p)\beta \exp(-\beta x),$$
  where $\delta(0)$ is the Dirac delta function and p and β are constants.

Parameters are chosen to match expected value and variance. (Andersen 2008)

Simulating S

Recall

$$\mathrm{d}S_t = \mu S_t\,\mathrm{d}t + \rho\sqrt{V_t}\,S_t\,\mathrm{d}W_t^1 + \sqrt{1-\rho^2}\sqrt{V_t}\,S_t\,\mathrm{d}W_t^2,$$
$$\mathrm{d}V_t = \kappa(\theta - V_t)\,\mathrm{d}t + \varepsilon\sqrt{V_t}\,\mathrm{d}W_t^1.$$

Given $V_{t_{n+1}} - V_{t_n}$ and $\int_{t_n}^{t_{n+1}} V_s\,\mathrm{d}s$, the log return $\log(S_{t_{n+1}}/S_{t_n})$ is Normal with mean

$$\big(\mu - \frac{\rho\kappa\theta}{\varepsilon}\big)(t_{n+1} - t_n) + \frac{\rho}{\varepsilon}\big(V_{t_{n+1}} - V_{t_n}\big) + \big(\frac{\rho\kappa}{\varepsilon} - \frac{1}{2}\big)\int_{t_n}^{t_{n+1}} V_s\,\mathrm{d}s,$$

and variance

$$\big(1-\rho^2\big)\int_{t_n}^{t_{n+1}} V_s\,\mathrm{d}s.$$

(Broadie & Kaya 2006)

Trapezoidal Rule

Task: Simulate $\left(V_{t_{n+1}} - V_{t_n}, \int_{t_n}^{t_{n+1}} V_s \,\mathrm{d}s\right)$

- Laplace transform of $\int_{t_n}^{t_{n+1}} V_s \,\mathrm{d}s$ given $V_{t_{n+1}}$ and V_{t_n} (Pitman & Yor 1982, Broadie & Kaya 2006)
- Representation as infinite sums and mixtures of independent Gamma-distributed random variables (Glasserman & Kim 2009)
- Trapezoidal rule (Andersen 2007): approximation of time integral by

$$\frac{1}{2}\big(V_{t_{n+1}} + V_{t_n}\big)\big(t_{n+1} - t_n\big).$$

Require martingale: $\mathrm{e}^{-\mu(t_{n+1}-t_n)} E\left[S_{t_{n+1}} \middle| (V_{t_n}, S_{t_n})\right] = S_{t_n}$
$\rightsquigarrow$ adjustment by multiplicative factor

$$\exp\big(K_0(t_{n+1} - t_n) + K_1 V_{t_n}\big).$$

Test Cases

	Case I	Case II
ϵ	1	0.9
κ	0.5	0.3
ρ	-0.9	-0.5
T	10	15
$V(0),\ \theta$	0.04	0.04
$4\kappa\theta/\varepsilon^2$	8/100	48/810

Table: Test cases from Andersen. In all cases $S(0) = 100$.

Test cases are "challenging and practically relevant"

- Case I representative of long-dated FX option market,
- Case II representative of long-dated interest-rate option market.

(Andersen 2008, p. 26.)

Numerical Results Case I: Error and Sdev

h	Andersen	Marsaglia
	Strike 100	
1	0.2211 (0.012)	-0.2374 (0.013)
1/2	0.1164 (0.013)	-0.0707 (0.013)
1/4	0.0143* (0.013)	-0.0440 (0.013)
1/8	-0.0277* (0.013)	-0.0050* (0.013)
1/16	0.0162* (0.013)	0.0019* (0.013)
	Strike 140	
1	-0.0883 (0.002)	-0.0283 (0.002)
1/2	-0.0274 (0.003)	-0.0121 (0.002)
1/4	-0.0013 (0.003)	-0.0048 (0.003)
1/8	0.0047 (0.003)	-0.0011 (0.003)
1/16	0.0018 (0.003)	0.0015 (0.003)
	Strike 60	
1	0.0317* (0.025)	-0.1234 (0.025)
1/2	0.0345* (0.025)	-0.0556* (0.025)
1/4	0.0111* (0.025)	-0.0388* (0.025)
1/8	0.0407* (0.025)	0.0120* (0.025)
1/16	0.0284* (0.025)	0.0003* (0.025)

Numerical Results Case II: Error and Sdev

h	Andersen	Marsaglia
	Strike 100	
1	-0.4833 (0.042)	-0.1404 (0.042)
1/2	-0.0400* (0.046)	-0.0264* (0.044)
1/4	-0.0231* (0.044)	0.0217* (0.048)
1/8	0.0807* (0.045)	-0.0553* (0.052)
1/16	-0.0026* (0.042)	0.0521* (0.046)
	Strike 140	
1	-0.3082 (0.036)	-0.0926* (0.036)
1/2	0.0515* (0.040)	0.0029* (0.037)
1/4	-0.0016* (0.038)	0.0207* (0.043)
1/8	0.0740* (0.039)	-0.0327* (0.047)
1/16	0.0069* (0.035)	0.0509* (0.040)
	Strike 60	
1	0.1180 (0.048)	-0.0379* (0.049)
1/2	0.1349 (0.052)	-0.0036* (0.050)
1/4	-0.0066* (0.050)	0.0290* (0.054)
1/8	0.0809* (0.052)	-0.0650* (0.058)
1/16	-0.0170* (0.049)	0.0492* (0.052)

Numerical Results

- Generalized Marsaglia method compares very favourably with Andersen's method in terms of efficiency (average CPU time over all steps): it is two times faster than Andersen's method in case 1 and uses 20% less CPU time in case 2.
- Convergence rate in Andersen's method unknown.
- Generalized Marsaglia method has advantage of simulating the chi-square distribution *exactly*.

Conclusion

- Derive representation of a chi-square random variable as sum of powers of independent generalized Gaussian random variables.
- Prove a new method – the generalized Marsaglia method – for sampling generalized Gaussian random variables.
- Establish a new method to sample a chi-square distributed random variable, and thus to simulate the Cox–Ingersoll–Ross model exactly and efficiently.
- Establish a new method to simulate the Heston volatility model in cases that are "challenging and practically relevant" (Andersen 2008 p. 26).
- Method is efficient, robust and and easy to implement.