Variation Swaps on Time-Changed Lévy Processes

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Robust pricing of derivatives

Underlying F. Some derivative contract pays Z_T, a function of F's path. Ways to find the contract's price $Z_0 = \mathbb{E} Z_T$:

- Specify a model for the underlying F. Compute

$$\text{“Parametric”}: \quad Z_0 = V(\text{model parameters})$$

- But we are skeptical of all models. Instead let us find g such that for all models in some universe, we have one of:

$$\text{“Nonparametric”}: \quad Z_0 = \mathbb{E} g(F_T)$$

$$\text{“Semiparametric”}: \quad Z_0 = V(\mathbb{E} g(F_T), \text{ subset of parameters})$$

$$\text{“Nonparametric bounds”}: \quad Z_0 \lesseqgtr \mathbb{E} g(F_T)$$

  where $\mathbb{E} g(F_T)$ is observable, given prices of options on F_T.

Note that semiparametric may be *more* robust than nonparametric.

Assumptions

- Work in $(\Omega, \mathcal{F}, \{\mathcal{F}_u\}, \mathbb{P})$, where $\mathbb{P}$ is martingale measure.
- Underlying F is a positive martingale, for example a forward/futures price, or (under zero rates) a share price.
- $Y_t := \log(F_t/F_0)$, the log-returns process.
- $[Y]$ denotes the quadratic variation of Y.
- (The floating leg of) a *continuously-sampled variance swap* pays

$$[Y]_T$$

  at expiry T.
- (The floating leg of) a *discretely-sampled variance swap* pays $\sum_{n=0}^{N-1}(Y_{t_{n+1}} - Y_{t_n})^2$ where $0 = t_0 < t_1 < \cdots < t_N = T$.

Variance swap valuation – standard approach

Neuberger (1990), Dupire (92), Carr-Madan (98), Derman et al (99).

- Let a *log contract* pay $-Y_T = -\log(F_T/F_0)$. Assume existence.
- Assume F is *continuous*
- Then variance swap value = value of *two log contracts*

$$\mathbb{E}[Y]_T = 2\mathbb{E}(-Y_T)$$

- Widely influential as a reference point for volatility traders
- This result is the basis for the CBOE's VIX index, and other indicators of options-implied expectations of realized variance (VXN, RVX, VSTOXX, VDAX-NEW, etc).
- Robust ("model-free") in that it assumes only the continuity of underlying paths. But empirically jump risk does exist.

Extensions

Our results (exact semi-parametric pricing formulas)

extend the standard theory as follows:

- Our earlier talk introduced jump risk into F dynamics.
- Generalize payoffs to G-*variation* and *share-weighted* G-variation. Instead of cumulating $(\mathrm{d}Y_t)^2$, let us cumulate $G(\mathrm{d}Y_t)$.

Our "meta"-results provide explanations of:

- Why does the standard theory work: Why do *log* contracts price variance swaps? Why *two* log contracts?
- Which variance-related contracts admit semi-parametric valuations that have become *easy* to solve by our methods? Which contracts are still *hard* to solve semi-parametrically?

Lévy processes

An adapted process $(X_u)_{u \geq 0}$ with $X_0 = 0$ is a Lévy process if:

- $X_v - X_u$ is independent of $\mathcal{F}_u$ for $0 \leq u < v$
- $X_v - X_u$ has same distribution as X_{v-u} for $0 \leq u < v$
- $X_v \to X_u$ in probability, as $v \to u$

Lévy -Khintchine: There exist $a \in \mathbb{R}$, $\sigma \geq 0$, and a *Lévy measure* ν, with $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}} (1 \wedge x^2) \nu(\mathrm{d}x) < \infty$, such that each X_t has CF

$$\mathbb{E} e^{izX_t} = e^{t\psi(z)}$$

where

$$\psi(z) := iaz - \frac{1}{2}\sigma^2 z^2 + \int_{\mathbb{R}} (e^{izx} - 1 - izx\mathbf{1}_{|x|\leq 1}) \nu(\mathrm{d}x)$$

Intuition: $\nu(A) = \mathbb{E}(\text{number of jumps of size} \in A, \text{ per unit time})$.

Time-changed exponential Lévy processes

A share price could be modeled by an exponential Lévy process

$$F_t = F_0 \exp(X_t)$$

Indeed, the case that $X_t = at + \sigma W_t$ gives GBM with drift.

But drawbacks:

- Today's return has same distribution as yesterday's.
- Today's return is independent of yesterday's.

Time-changed exponential Lévy processes

- Let X be a Lévy process such that $\mathbb{E}e^{X_1} < \infty$.

  Let $X'_u := X_u - u \log \mathbb{E}e^{X_1}$, so that $e^{X'}$ is a martingale.
- Let the time change $\{\tau_t\}_{t\in[0,T]}$ be an increasing continuous family of stopping times.

  So τ is a stochastic "clock" that measures "business time":

  $$\text{Calendar time } t \quad \longleftrightarrow \quad \text{Business time } \tau_t$$

  We do *not* assume that τ and X are independent.
- Assume $Y_t = X'_{\tau_t}$ and $F_t = F_0 \exp(Y_t)$.

  The time-changed Lévy process Y can exhibit stochastic volatility, stochastic jump intensity, volatility clustering, and "leverage" effects.
- By DDS, this family includes all positive continuous martingales.

Variance swaps on time-changed Lévy processes

Our earlier work introduced jumps:

- Variance swaps still admit pricing in terms of log contracts.
- However the correct number of log contracts may not be 2.
- The correct variance swap *multiplier* depends only on the dynamics of the Lévy driver X, not on the time change τ.
- Explicit formula for multiplier:

$$Q^{X,G} = \frac{\sigma^2 + \int x^2 \mathrm{d}\nu(x)}{\sigma^2/2 + \int (e^x - 1 - x)\mathrm{d}\nu(x)}.$$

- Whether multiplier is greater or less than 2 depends on skewness of Lévy measure.
- Effect of discrete sampling

G-variation of a semimartingale Y

For a general semimartingale Y, we define the G-variation of Y. Let

$$G(x) = \alpha|x| + \gamma x^2 + o(x^2)$$

be continuous, where $\alpha \geq 0$ and γ are constants and:

- Either $\alpha = 0$ or Y has finite variation. If the latter, then let $Y_t^{\mathrm{d}} := Y_t - \sum_{0<s\leq t} \Delta Y_s$, and let $\mathrm{TV}(Y^{\mathrm{d}})$ be total variation of Y^{d}.
- The $o(x^2)$ is for $x \to 0$. It can be relaxed if $Y^{\mathrm{c}} = 0$, where Y^{c} is the continuous local martingale part of Y.

Then define the (continuously-sampled) G-variation of Y by

$$V_t^{Y,G} := \alpha \mathrm{TV}(Y^{\mathrm{d}})_t + \gamma [Y^{\mathrm{c}}]_t + \sum_{0<s\leq t} G(\Delta Y_s)$$

(where $\alpha\mathrm{TV} := 0$ if $\alpha = 0$).

G-variation of a semimartingale Y

More generally, let

$$G(x) = \alpha|x| + \gamma x^2 + g(x)$$

where g satisfies any of

$$g(x) = o(x^2),$$

$$g(x) = O(|x|^r) \text{ and } r \in I \cap (1, 2] \text{ and } Y^{\mathrm{c}} = 0$$

$$g(x) = O(|x|^r) \text{ and } r \in I \cap (0, 1] \text{ and } Y^{\mathrm{d}} = Y^{\mathrm{c}} = 0$$

where

$$I := \{r \geq 0 : \int_{(0,t]\times\mathbb{R}} (|x|^r \wedge 1)\mathrm{d}\nu_Y < \infty \text{ for all } t > 0\}$$

where ν_Y denotes the jump compensator of Y

Motivation for definition of G-variation

For sampling interval Δ_n, let us define the *discretely-sampled* *G-variation* of Y by

$$V^{Y,G}(n)_T := \sum_{j=1}^{\lfloor T/\Delta_n \rfloor} G(Y_{j\Delta_n} - Y_{(j-1)\Delta_n})$$

As $n \to \infty$, if $\Delta_n \to 0$, we have Skorokhod convergence in probability

$$V^{Y,G}(n) \longrightarrow V^{Y,G}.$$

This motivates our definition of $V^{Y,G}$, and justifies referring to it as "continuously-sampled" G-variation. Intuition:

$$V_T^{Y,G} = \int_0^T G(\mathrm{d}Y_t)$$

Examples of G-variation swaps

Canonical example is a variance swap: $G(x) = x^2$. Other examples:

- Total variation swap

$$G(x) = |x|$$

- Simple-returns variance swap

$$G(x) = (e^x - 1)^2$$

- Moment swap, for integer $p > 1$

$$G(x) = x^p$$

- Absolute moment swap, for real $p > 1$

$$G(x) = |x|^p$$

Examples of G-variation swaps

- Capped-movement versions of above: Replace G with

$$G(\min(\max(x, a), b))$$

where $-\infty \le a < b \le \infty$. Example: Down *semivariance*, where

$$G(x) := (x \wedge 0)^2,$$

is a statistic of interest to portfolio managers.

- Capped-G versions of above: Replace G with

$$\min(G(x), M)$$

Example: Capped variance

$$G(x) = x^2 \wedge M$$

limits the liability of variance sellers.

Variation swaps on time-changed Lévy processes

We show that:

- G-variation swaps still admit pricing in terms of log contracts.
- However the correct number of log contracts may not be 2.
- The correct variation swap *multiplier* depends only on G and the dynamics of the Lévy driver X, not on the time change τ.

The multiplier

- Let X be a nondeterministic Lévy process, with $\mathbb{E}e^{X_1} < \infty$ and $\mathbb{E}|X|_1 < \infty$ and $\int G \mathrm{d}\nu < \infty$. Define the multiplier of (X, G) by

$$Q^{X,G} := \frac{\mathbb{E}V_1^{X',G}}{-\mathbb{E}X_1'} = \frac{\mathbb{E}V_1^{X',G}}{\log \mathbb{E}e^{X_1} - \mathbb{E}X_1}$$

- Proposition: Let X have generating triplet (A, σ^2, ν). Then

$$Q^{X,G} = \frac{\left|\alpha\sigma^2/2 + \int \alpha(e^x - 1)\mathrm{d}\nu(x)\right| + \gamma\sigma^2 + \int G(x)\mathrm{d}\nu(x)}{\sigma^2/2 + \int (e^x - 1 - x)\mathrm{d}\nu(x)}.$$

Proof: Denominator is sum of $-\mathbb{E}X_1 = -A - \int_{|x|\geq 1} x\nu(\mathrm{d}x)$ and

$$\log \mathbb{E}e^{X_1} = A + \sigma^2/2 + \int (e^x - 1 - x\mathbf{1}_{|x|\leq 1})\nu(\mathrm{d}x)$$

Numerator is sum of "(ex-jump) drift", "Brownian", and "jump" contributions to $\mathbb{E}V_1^{X',G}$.

Pricing variation swaps on time-changed Lévy processes

Proposition

If $\mathbb{E}\tau_T < \infty$ *then*

$$\mathbb{E}V_T^{Y,G} = Q^{X,G}\mathbb{E}(-Y_T).$$

Hence the variation swap and $Q^{X,G}$ *log contracts have the same value.*

Proof.

$V_u^{X',G} + Q^{X,G}X'_u$ is a Lévy martingale, so by Wald's equation

$$\mathbb{E}(V_{\tau_T}^{X',G} + Q^{X,G}X'_{\tau_T}) = 0.$$

By τ continuity, $\mathbb{E}V_T^{Y,G} = \mathbb{E}V_T^{X'_{\tau_\cdot},G} = \mathbb{E}V_{\tau_T}^{X',G} = Q^{X,G}\mathbb{E}(-Y_T)$. □

Proposition: $\mathbb{E}V_T^{Y,G} = Q^{X,G}\mathbb{E}(-Y_T)$

Idea of proof: For all fixed times u, by Lévy property of X and $V^{X',G}$,

$$\mathbb{E}V_u^{X',G} = Q^{X,G}\mathbb{E}(-X_u).$$

Replace u with τ_T, by a form of the optional stopping theorem:

$$\mathbb{E}V_{\tau_T}^{X',G} = Q^{X,G}\mathbb{E}(-X_{\tau_T}).$$

Exchange variation operator and time-change, by continuity of τ:

$$\mathbb{E}V_T^{X'_{\tau.},G} = Q^{X,G}\mathbb{E}(-X_{\tau_T}).$$

as claimed.

In this setting, jumps arise from X jumping, not from clock jumping (although we allow the clock *rate* to jump).

Example: Time-changed geometric Brownian motion

Let X be Brownian motion and $G(x) = x^2$. Then

$$Q^{X,G} = \frac{\mathbb{E}[X]_1}{-\mathbb{E}X_1'} = \frac{1}{1/2} = 2$$

This recovers the 2 multiplier for all positive continuous martingales.

Example: Time-changed fixed-size jump diffusion

Let X have Brownian variance σ^2 and Lévy measure

$$\lambda_1 \delta_{c_1} + \lambda_2 \delta_{c_2}$$

where δ_c denotes a point mass at c, and $c_1 > 0$ and $c_2 < 0$. Then

$$Q^{X,G} = \frac{\alpha|\lambda_1(e^{c_1} - 1) + \lambda_2(e^{c_2} - 1)| + \gamma\sigma^2 + \lambda_1 G(c_1) + \lambda_2 G(c_2)}{\sigma^2/2 + \lambda_1(e^{c_1} - 1 - c_1) + \lambda_2(e^{c_2} - 1 - c_2)}.$$

In particular, consider $G(x) = x^2$.

Third-order Taylor expansion in (c_1, c_2) about $(0, 0)$, if $\sigma \neq 0$:

$$Q^{X,G} \approx 2 - \frac{2\lambda_1}{3\sigma^2} c_1^3 + \frac{2\lambda_2}{3\sigma^2} |c_2|^3,$$

increasing in absolute down-jump size, decreasing in up-jump size.

Time-changed Kou double-exponential jump-diffusion

Let X have Brownian variance σ^2 and Lévy density

$$\nu(x) = \lambda_1 a_1 e^{-a_1|x|}\mathbf{1}_{x>0} + \lambda_2 a_2 e^{-a_2|x|}\mathbf{1}_{x<0}$$

where $a_1 \geq 1$ and $a_2 > 0$. So up-jumps have mean size $1/a_1$, down-jumps have mean absolute size $1/a_2$.

For $G(x) = x^2$,

$$Q^{X,G} = \frac{\sigma^2 + 2\lambda_1/a_1^2 + 2\lambda_2/a_2^2}{\sigma^2/2 + \lambda_1/(a_1-1) - \lambda_2/(a_2+1) - \lambda_1/a_1 + \lambda_2/a_2}.$$

Third-order Taylor expansion in $(1/a_1, 1/a_2)$ about $(0,0)$, if $\sigma \neq 0$:

$$Q^{X,G} \approx 2 - \frac{4\lambda_1/\sigma^2}{a_1^3} + \frac{4\lambda_2/\sigma^2}{a_2^3},$$

Example: Time-changed extended CGMY

Let X have the extended CGMY Lévy density

$$\nu(x) = \frac{C_n}{|x|^{1+Y_n}} e^{-G|x|} \mathbf{1}_{x<0} + \frac{C_p}{|x|^{1+Y_p}} e^{-M|x|} \mathbf{1}_{x>0},$$

where $C_p, C_n > 0$ and $G, M > 0$, and $Y_p, Y_n < 2$. For $G(x) = x^2$, $Q^{X,G} =$

$$\frac{-C_n\Gamma(2-Y_n)G^{Y_n-2} - C_p\Gamma(2-Y_p)M^{Y_p-2}}{C_n\Gamma(-Y_n)[G^{Y_n}-(G+1)^{Y_n}+Y_nG^{Y_n-1}]+C_p\Gamma(-Y_p)[M^{Y_p}-(M-1)^{Y_p}-Y_pM^{Y_p-1}]}$$

Expanding the denominator in $1/G$ and $1/M$,

$$Q^{X,G} \approx 2 \times \frac{G^{Y_n-2} + \rho M^{Y_p-2}}{G^{Y_n-2}(1 - \frac{2-Y_n}{3G} + \ldots) + \rho M^{Y_p-2}(1 + \frac{2-Y_p}{3M} + \ldots)}.$$

where $\rho := C_p\Gamma(2-Y_p)/(C_n\Gamma(2-Y_n))$.

Example: Time-changed VG

The Variance Gamma model takes $Y = 0$.

$$\nu(x) = \frac{C}{|x|} e^{-G|x|} \mathbf{1}_{x<0} + \frac{C}{|x|} e^{-M|x|} \mathbf{1}_{x>0}$$

For $G(x) = x^2$, its multiplier is

$$\begin{aligned} Q^{X,G} &= \frac{1/G^2 + 1/M^2}{-\log(1+1/G) + 1/G - \log(1-1/M) - 1/M} \\ &\approx 2 \times \frac{G^{-2} + M^{-2}}{G^{-2}(1 - \frac{2}{3G} + \ldots) + M^{-2}(1 + \frac{2}{3M} + \ldots)}. \end{aligned}$$

Note the sign asymmetry between the $-\frac{2}{3G}$ and the $+\frac{2}{3M}$.

Example: Time-changed normal inverse Gaussian (NIG)

Let X have no Brownian component. Let X have Lévy density

$$\nu(x) = \frac{\delta\alpha}{\pi}\frac{\exp(\beta x)K_1(\alpha|x|)}{|x|},$$

where $\delta > 0$, $\alpha > 0$, $|\beta| < \alpha$, and K_1 = modified Bessel function of the second kind and order 1. Then X has cumulant transform

$$\kappa(z) = \log \mathbb{E}e^{zX_1} = \gamma z + \delta(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + z)^2}),$$

for some γ that we need not specify. Then for $G(x) = x^2$,

$$Q^{X,G} = \frac{\kappa''(0)}{\kappa(1) - \kappa'(0)} = \frac{\alpha^2/(\alpha^2 - \beta^2)}{\alpha^2 - \beta^2 - \beta - \sqrt{(\alpha^2 - \beta^2)(\alpha^2 - (\beta + 1)^2)}}.$$

Small jump-size limit: take $\alpha \to \infty$. Expanding in $1/\alpha$,

$$Q^{X,G} \approx 2 - \frac{4\beta + 1}{2\alpha^2}.$$

which is decreasing in β, the parameter which controls the “tilt”.

Definition

Define the dual or share-weighted G-variation of Y by

$$\tilde{V}_t^{Y,G} := \int_0^t e^{Y_s} \mathrm{d}V_s^{Y,G} = \int_0^t \frac{F_s}{F_0} \mathrm{d}V_s^{Y,G}$$

where the integrals are pathwise Riemann-Stieltjes.

Share-weighted variation swaps, which pay $\tilde{V}_t^{Y,G}$, confer variation exposure proportional to underlying level F. Motivations:

- Investor may be bullish
- Investor may have view that the market's downward implied volatility skew is too steep.
- Investor may be seeking to hedge variation exposure that grows as Y increases, e.g. in dispersion trading
- Investor may wish to trade single-stock variance without caps

Examples of share-weighted variation swaps

- Share-weighted counterparts exist, for each example of G.
- Canonical example is the *gamma swap*: $G(x) = x^2$

  Standard theory: Gamma swap has same value as 2 contracts on

$$(F_T/F_0)\log(F_T/F_0).$$

- *Pre-jump* share-weighted G-variation swap uses modified weights:

$$\int_0^t \frac{F_{s-}}{F_0} \mathrm{d}V_s^{Y,G}$$

  This is equivalent to using share-weighted variation with respect to the function $e^{-x}G(x)$.

Pricing share-weighted variation swaps

Proposition

Again let $G(x) = \alpha|x| + \gamma x^2 + g(x)$. *Under integrability conditions,*

$$\mathbb{E}\tilde{V}_T^{Y,G} = \tilde{Q}^{X,G}\mathbb{E}((F_T/F_0)\log(F_T/F_0)).$$

where the dual multiplier

$$\tilde{Q}^{X,G} := \frac{\left|\alpha\sigma^2/2 + \int \alpha(1-e^x)\mathrm{d}\nu(x)\right| + \gamma\sigma^2 + \int e^x G(x)\mathrm{d}\nu(x)}{\sigma^2/2 + \int(1 - e^x + xe^x)\mathrm{d}\nu(x)}.$$

Proof.

Change to "share measure" $\tilde{\mathbb{P}}$ where $\mathrm{d}\tilde{\mathbb{P}}_u/\mathrm{d}\mathbb{P}_u = \exp X'_u$.

Apply unweighted result to $\tilde{X} := -X$ and $\tilde{G}(x) := G(-x)$. □

Share-weighted variation swaps, with jump risk

We have shown that, for generalized variation,

in the presence of jump risk,

- Share-weighted G-variation swaps still admit pricing in terms of $F \log F$ contracts.
- However the correct number of $F \log F$ contracts may not be 2.
- The correct share-weighted variation swap *multiplier* depends only on G and the dynamics of the Lévy driver X, not on the time change τ.

Skewness impact depends on the contract

- $Q^{X,G} - 2$ has same sign as $\mathbb{E}V_1^{X',G} - 2\mathbb{E}(-X_1') =$

$$\int \Big(-\frac{x^3}{3} - \frac{x^4}{12} + O(x^5) \Big) \mathrm{d}\nu(x) \qquad \text{for } G(x) = x^2$$

$$\int \Big(\frac{2x^3}{3} + \frac{x^4}{2} + O(x^5) \Big) \mathrm{d}\nu(x) \qquad \text{for } G(x) = (e^x - 1)^2$$

- $\tilde{Q}^{X,G} - 2$ has same sign as $\mathbb{E}\tilde{V}_1^{X',G} - 2\mathbb{E}(X_1' e^{X_1'}) =$

$$\int \Big(\frac{x^3}{3} + \frac{x^4}{6} + O(x^5) \Big) \mathrm{d}\nu(x) \qquad \text{for } G(x) = x^2$$

$$\int \Big(-\frac{2x^3}{3} - \frac{x^4}{4} + O(x^5) \Big) \mathrm{d}\nu(x) \qquad \text{for } G(x) = e^{-x} x^2$$

$$\int \Big(\frac{4x^3}{3} + \frac{11x^4}{6} + O(x^5) \Big) \mathrm{d}\nu(x) \qquad \text{for } G(x) = (e^x - 1)^2$$

$$\int \Big(\frac{x^3}{3} + \frac{x^4}{3} + O(x^5) \Big) \mathrm{d}\nu(x) \qquad \text{for } G(x) = e^{-x}(e^x - 1)^2$$

Multipliers of empirically calibrated processes

		Variance			Simple variance			Third moment		
X	Data	1	F_t	F_{t-}	1	F_t	F_{t-}	1	F_t	F_{t-}
CGMY	Jun	2.37	1.81	2.70	1.62	1.53	1.85	-1.85	-0.42	-2.11
CGMY	Sep	2.17	1.87	2.33	1.76	1.62	1.89	-0.61	-0.33	-0.65
CGMY	Dec	2.13	1.88	2.27	1.78	1.63	1.89	-0.45	-0.31	-0.48
VG	Jun	2.10	1.91	2.20	1.83	1.69	1.92	-0.32	-0.25	-0.34
VG	Sep	2.09	1.92	2.18	1.84	1.72	1.92	-0.28	-0.23	-0.30
VG	Dec	2.10	1.91	2.21	1.82	1.67	1.91	-0.33	-0.27	-0.34
NIG	Jun	2.12	1.89	2.25	1.79	1.63	1.90	-0.39	-0.31	-0.42
NIG	Sep	2.11	1.90	2.22	1.81	1.66	1.91	-0.35	-0.28	-0.36
NIG	Dec	2.10	1.90	2.21	1.82	1.67	1.91	-0.33	-0.27	-0.35

Perfect hedging with two jump sizes

Let X have jump sizes $c_1 > 0$ and $c_2 < 0$ and zero Brownian part, and piecewise constant paths:

$$\nu = \lambda_1 \delta_{c_1} + \lambda_2 \delta_{c_2}$$

where $\lambda_1 := (1 - e^{c_2})\lambda$ and $\lambda_2 := (e^{c_1} - 1)\lambda$, for arbitrary $\lambda > 0$.

Then

$$Q^{X,G} \log(F_0/F_T) + \int_0^T \frac{q^{X,G}}{F_{t-}} \mathrm{d}F_t = V_T^{Y,G}.$$

where $q^{X,G} := (c_2 G(c_1) - c_1 G(c_2))/(c_2(e^{c_1} - 1) + c_1(1 - e^{c_2}))$.

So replicate $V_T^{Y,G}$ by holding:

- $Q^{X,G}$ log contracts, statically
- $q^{X,G}/F_{t-}$ shares, dynamically

Intuition

Does decreasing the sampling frequency tend to increase or decrease the expectation of realized variance? Intuition:

Consider 1 long sampling period vs. 2 shorter sampling periods.

With log-returns of R_1 and R_2 in the two periods,

the more-frequently-sampled realized variance is

$$R_1^2 + R_2^2$$

The less-frequently-sampled realized variance is

$$(R_1 + R_2)^2 = R_1^2 + R_2^2 + 2R_1R_2$$

So coarser sampling adds

$$2R_1R_2$$

to the realized variance.

Intuition

- The expected impact of less sampling is

$$2\mathbb{E}(R_1R_2) = 2\mathbb{E}R_1\mathbb{E}R_2 + 2\mathrm{Cov}(R_1, R_2)$$

- If R would be martingale increments, $\mathbb{E}(R_1R_2)$ would vanish. Indeed, realized variance of a martingale M is perfectly replicable, continuously

$$M_T^2 = M_0^2 + \int_0^T 2M_{t-}\mathrm{d}M_t + [M]_T$$

or discretely

$$M_{t_N}^2 = M_0^2 + \sum 2M_{t_n}(\Delta M_{t_n}) + \sum(\Delta M_{t_n})^2$$

- But due to taking logs, $\mathbb{E}(R_1R_2) > 0$ typically.

Discrete sampling

Let $0 = t_0 < t_1 < \cdots < t_N = T$. Write $\Delta_n Z := Z_{t_{n+1}} - Z_{t_n}$.

If $\mathbb{E}\tau_T < \infty$ and τ and X are independent then

$$\mathbb{E}\sum_{n=0}^{N-1}(\Delta_n Y)^2 = \mathbb{E}[Y]_T + \sum_{n=0}^{N-1}(\mathbb{E}\Delta_n Y)^2 + \sum_{n=0}^{N-1}\mathrm{Var}(\mathbb{E}(\Delta_n Y|\tau)).$$

Proof: $M_t := Y_t - \tau_t \mathbb{E}X_1$ is a martingale. Sum the following over n:

$$\begin{aligned}
\mathbb{E}(\Delta[Y]) &= \mathbb{E}(\Delta[M]) = \mathbb{E}(\Delta M)^2 = \mathbb{E}(\Delta Y - (\Delta\tau)\mathbb{E}X_1)^2 \\
&= \mathbb{E}(\Delta Y - \mathbb{E}(\Delta Y|\tau))^2 = \mathbb{E}(\mathrm{Var}(\Delta Y|\tau)) \\
&= \mathrm{Var}(\Delta Y) - \mathrm{Var}(\mathbb{E}(\Delta Y|\tau)) \\
&= \mathbb{E}(\Delta Y)^2 - (\mathbb{E}\Delta Y)^2 - \mathrm{Var}(\mathbb{E}(\Delta Y|\tau)).
\end{aligned}$$

Hence discrete sampling increases variance swap values. Premium depends on squared spreads of log contracts, and $\mathrm{Var}(\mathbb{E}(\Delta Y|\tau))$.

Why does standard theory work

- Why do *log* contracts price variance swaps?

  Because, if F is an exponential Lévy process, then

  $$\log F \qquad \text{and} \qquad [\log F]$$

  are both Lévy processes.

  So the ratio Q of their "drifts" gives their relative price.

  This property survives under continuous time change – and such time changes generate all continuous positive martingales.

- Why *two* log contracts?

  Because $-\log(\text{GBM})$ has drift $1/2$. So the drift ratio is 2.

Extension to jumps

The drift-ratio reasoning still holds, but with a different ratio.

Variance swap value = a multipler (Q) times log contract value.

True for all time-changed Lévy processes. Arbitrary stochastic clock, arbitrary correlation. The Q does not depend on the clock.

- For continuous underlying paths, $Q = 2$.
- In the presence of negatively skewed jump risk,

$$Q > 2$$

  In that case, quotations based on a 2 multiple (including VIX) would underprice the continuously-sampled variance, and typically furthermore underprice the discretely-sampled variance.

Extension to other contracts

Let $G(x) = \alpha|x| + \gamma x^2 + o(x^2)$.

- By *same* techniques, we price a G-*variation* swap which pays

$$V_T = \alpha \mathrm{TV}(Y^{\mathrm{d}})_T + \gamma [Y^{\mathrm{c}}]_T + \sum_{0<s\leq T} G(\Delta Y_s)$$

  (subject to conditions on G, Y), because V_t is Lévy if Y is.

- By *same* techniques, we price *share-weighted* G-variation:

$$\tilde{V}_t^{Y,G} := \int_0^t \frac{F_s}{F_0} \mathrm{d}V_s^{Y,G}$$

  in terms of an $F_T \log F_T$ contract, via measure change.

- Under further conditions, can price *volatility derivatives* paying

$$h([Y]_T).$$

  Different techniques needed, because $h([Y]_t)$ may not be Lévy