Processes of Class (Σ), Last Passage Times and Drawdowns

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joint work with Ashkan Nikeghbali and Eckhard Platen

Processes of class (Σ) were introduced in

M. Yor (1979).
Les inégalités de sous-martingales comme conséquence de la relation de domination. Stochastics **3**(1).

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space

Definition

We say (X_t) is a process of class (Σ) if $X_t = N_t + A_t$, where

(1) (N_t) is a cadlag local martingale

(2) A_t is a continuous adapted finite variation process starting at 0

(3) $\int_0^t 1_{\{X_u \neq 0\}} dA_u = 0$ for all $t \geq 0$

We say (X_t) is of class (ΣD) if it is of class (Σ) and of class (D).

$L := \sup\{t : X_t = 0\}$ with the convention $\sup \emptyset = 0$.

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(2) If (M_t) is a continuous local martingale, then

$$|M_t| = |M_0| + \int_0^t \mathrm{sign}(M_u) dM_u + l_t,$$

$$M_t^+ = M_0^+ + \int_0^t 1_{\{M_u > 0\}} dM_u + \frac{1}{2} l_t \text{ and}$$

$$M_t^- = M_0^- + \int_0^t 1_{\{M_u \leq 0\}} dM_u + \frac{1}{2} l_t$$

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(3) If (M_t) is a local martingale such that

$$\overline{M}_t := \sup_{u \leq t} M_u \quad \text{is continuous, then}$$

$$\overline{M}_t - M_t = (M_0 - M_t) + (\overline{M}_t - M_0)$$

is of class (Σ).

Lemma

Let (X_t) be of class (ΣD). Then (N_t) is a uniformly integrable martingale and (A_t) of totally integrable variation. In particular,

$$N_t \to N_\infty, \quad A_t \to A_\infty, \quad X_t \to X_\infty$$

almost surely and in L^1.

Proof

X_t^+ and X_t^- are submartingales of class (D), and the lemma follows from the Doob-Meyer Theorem. □

Lemma

Let (X_t) be of class (Σ) and $f : \mathbb{R} \to \mathbb{R}$ differentiable. Then

$f(A_t)X_t$ is of class (Σ) with decomposition

$$f(A_t)X_t = f(0)X_0 + \int_0^t f(A_u)dN_u + F(A_t),$$

where

$$F(x) = \int_0^x f(y)dy.$$

In particular, if $f(A_t)X_t$ is of class (D), then

$$\begin{aligned} f(A_t)X_t - F(A_t) &= f(0)X_0 + \int_0^t f(A_u)dN_u \\ &= \mathsf{E}\left[f(A_\infty)X_\infty - F(A_\infty) \mid \mathcal{F}_t\right] \end{aligned}$$

Proof

$$\begin{aligned} d(f(A_t)X_t) &= f'(A_t)X_t dA_t + f(A_t)dN_t + f(A_t)dA_t \\ &= f(A_t)dN_t + f(A_t)dA_t. \end{aligned}$$

$$\int_0^t 1_{\{f(A_u)X_u \neq 0\}} dF(A_u) = \int_0^t 1_{\{f(A_u)X_u \neq 0\}} f(A_u)dA_u = 0.$$

$\square$

The transformation

$$X_t \to f(A_t)X_t$$

is inspired by the Azéma–Yor solution of the Skorokhod embedding problem.

See also Carraro–El Karoui–Obloj (2009)

Theorem

Let (X_t) be of class $(\Sigma\mathrm{D})$. Then

$$X_t = \mathsf{E}\left[X_\infty 1_{\{L \le t\}} \mid \mathcal{F}_t\right], \quad t \ge 0.$$

Proof

$$X_t 1_{\{L\le t\}} = \left\{ \begin{array}{ll} X_\infty & \text{if } L \le t \\ 0 & \text{if } L > t \end{array} \right\} = X_{d_t},$$

where

$$d_t = \inf\{u > t : X_u = 0\}, \quad \inf \emptyset = \infty.$$

$$\begin{aligned} & \mathsf{E}\left[X_\infty 1_{\{L\le t\}} \mid \mathcal{F}_t\right] = \mathsf{E}\left[X_{d_t} \mid \mathcal{F}_t\right] \\ = \; & \mathsf{E}\left[N_{d_t} + A_{d_t} \mid \mathcal{F}_t\right] = N_d + A_t = X_t \end{aligned}$$

□

Corollary **(Madan–Roynette–Yor, 2008)**

Let (M_t) be a non-negative local martingale with no negative jumps. Let $K \in \mathbb{R}_+$ and

$$g^K = \sup\{t : M_t \geq K\}$$

Then

$$(K - M_t)^+ = \mathsf{E}\left[(K - M_\infty)^+ 1_{\{g^K \leq t\}} \mid \mathcal{F}_t\right].$$

In particular, if $M_t \to 0$ a.s., then

$$\mathsf{E}\left[(K - M_t)^+\right] = K\mathbb{P}[g^K \leq t].$$

Corollary

Let (M_t) be a non-negative local martingale with no positive jumps such that $M_t \to 0$ a.s. Define

$$T_{\max} = \sup\left\{t \geq 0 : M_t = \overline{M}_\infty\right\}.$$

Then

$$\mathbb{P}[T_{\max} \leq t] = \mathsf{E}\left[1 - \frac{M_t}{\overline{M_t}}\right] = \mathsf{E}\left[\log(\overline{M}_t)\right] - \mathsf{E}\left[M_0\right].$$

Proof

$$X_t = 1 - \frac{M_t}{\overline{M_t}} = \frac{\overline{M}_t - M_t}{\overline{M}_t} \quad \text{is of class } (\Sigma)$$

with $L = T_{\max}$. □

Theorem

Let (X_t) be a non-negative process of class (Σ) and $f : \mathbb{R}_+ \to \mathbb{R}_+$ differentiable such that $f(A_t)X_t$ is of class (D) and $f(A_t)X_t \to 1$ a.s. Denote

$$F(x) = \int_0^x f(y)dy$$

Then for all bounded Borel functions $h : \mathbb{R} \to \mathbb{R}$ and every stopping time T,

$$\begin{aligned} & \mathsf{E}\left[h(A_\infty) \mid \mathcal{F}_T\right] \\ = \ & h(0)f(0)X_0 + h^F(0)(1 - f(0)X_0) \\ & + \int_0^T (h - h^F)(A_u)f(A_u)dN_u \\ = \ & h(A_T)f(A_T)X_T + h^F(A_T)(1 - f(A_T)X_T), \end{aligned}$$

where

$$h^F(x) = e^{F(x)} \int_x^\infty h(y)e^{-F(y)}dF(y), \quad x \geq 0.$$

In particular, the conditional law of A_∞ is given by

$$\begin{aligned} & \mathbb{P}[A_\infty > x \mid \mathcal{F}_T] \\ = \ & 1_{\{A_T > x\}} + 1_{\{A_T \leq x\}}(1 - f(A_T)X_T)e^{F(A_T) - F(x)}. \end{aligned}$$

Let (Y_t) be a diffusion of the form

$$dY_t = \mu(Y_t)dt + \sigma(Y_t)dB_t, \quad Y_0 = y_0.$$

Consider a function $\lambda : [y_0, \infty) \to \mathbb{R}$ and define

$$T_\lambda = \inf\left\{t \geq 0 : Y_t = \lambda(\overline{Y_t})\right\}$$

Set

$$\gamma(x) = 2\int_{y_0}^{x} \frac{\mu(y)}{\sigma^2(y)}dy \quad \text{and} \quad s(x) = \int_{y_0}^{x} e^{-\gamma(y)}dy.$$

Then $M_t = s(Y_t)$ is a local martingale

Corollary (Lehocky, 1977)

$$\mathbb{P}[\overline{Y}_{T_\lambda} > x] = \exp\left(-\int_{y_0}^{x} \frac{e^{-\gamma(y)}dy}{\int_{\lambda(y)}^{y} e^{-\gamma(z)}dz}\right) \quad \text{for } x \geq y_0,$$

Let (M_t) be a continuous local martingale starting at $m \in \mathbb{R}_+$. Let $\lambda : [m, \infty) \to \mathbb{R}$ such that $\lambda(x) < x$. Define

$$T_\lambda = \inf\left\{t : M_t = \lambda(\overline{M}_t)\right\} = \inf\left\{t : \frac{\overline{M}_t - M_t}{\overline{M}_t - \lambda(\overline{M}_t)} = 1\right\}$$

and

$$\Lambda(x) = \int_0^x \frac{1}{y - \lambda(y)} dy.$$

Theorem

For each Borel function $h : \mathbb{R} \to \mathbb{R}$ and stopping time $T \leq T_\lambda$,

$$\begin{aligned} & \mathsf{E}\left[h(\overline{M}_{T_\lambda}) \mid \mathcal{F}_T\right] \\ = \; & h^\Lambda(m) + \int_0^T \frac{h^\Lambda(\overline{M}_u) - h(\overline{M}_u)}{\overline{M}_u - \lambda(\overline{M}_u)} dM_u \\ = \; & \frac{h(\overline{M}_T)(\overline{M}_T - M_T) + h^\Lambda(\overline{M}_T)(M_T - \lambda(\overline{M}_T))}{\overline{M}_T - \lambda(\overline{M}_T)}, \end{aligned}$$

where

$$h^\Lambda(x) = e^{\Lambda(x)} \int_x^\infty h(y) e^{-\Lambda(y)} d\Lambda(y), \qquad x \geq m.$$

Special Cases

Drawdown: $DD_t = \overline{M_t} - M_t$

Relative Drawdown: $rDD_t = \frac{\overline{M_t} - M_t}{\overline{M_t}}$

Consider triggers of the form:

1. Stop-loss trigger $T_c = \inf \{t : M_t = c\}$, $c < M_0$
2. Drawdown trigger $T_c = \inf \{t : DD_t = c\}$, $c > 0$
3. Relative drawdown trigger $T_c = \inf \{t : rDD_t = c\}$, $c > 0$

Stop-loss trigger

$T_c = \inf\{t : M_t = c\}$ for some $c < M_0$.

$$\begin{aligned} & \mathsf{E}\left[h(\overline{M}_{T_c}) \mid \mathcal{F}_{t\wedge T_c}\right] \\ = \; & h^{\wedge}(m) + \int_0^{t\wedge T_c} \frac{h^{\wedge}(\overline{M}_u) - h(\overline{M}_u)}{\overline{M}_u - c} dM_u \\ = \; & \frac{h(\overline{M}_{t\wedge T_c}) DD_{t\wedge T_c} + h^{\wedge}(\overline{M}_{t\wedge T_c})(M_{t\wedge T_c} - c)}{\overline{M}_{t\wedge T_c} - c}, \end{aligned}$$

where

$$h^{\wedge}(x) = (x - c) \int_x^{\infty} \frac{h(y)}{(y-c)^2} dy.$$

Drawdown trigger

$T_c = \inf\{t : DD_t = c\}$ for some $c > 0$.

$$
\begin{aligned}
& \mathsf{E}\left[h(\overline{M}_{T_c}) \mid \mathcal{F}_{t\wedge T_c}\right] \\
= \;& h^{\wedge}(m) + \frac{1}{c}\int_0^{t\wedge T_c} (h^{\wedge}(\overline{M}_u) - h(\overline{M}_u))dM_u \\
= \;& h(\overline{M}_{t\wedge T_c})\frac{DD_{t\wedge T_c}}{c} + h^{\wedge}(\overline{M}_{t\wedge T_c})\left(1 - \frac{DD_{t\wedge T_c}}{c}\right).
\end{aligned}
$$

for

$$
h^{\wedge}(x) = \frac{1}{c}e^{x/c}\int_x^{\infty} h(y)e^{-y/c}dy.
$$

Relative drawdown trigger

$T_c = \inf\{t : rDD_t = c\}$ for some $c < M_0$.

$$\begin{aligned}
& \mathsf{E}\left[h(\overline{M}_{T_c}) \mid \mathcal{F}_{t\wedge T_c}\right] \\
= \;& h^{\wedge}(m) + \int_0^{t\wedge T_c} \frac{h^{\wedge}(\overline{M}_u) - h(\overline{M}_u)}{c\overline{M}_u} dM_u \\
= \;& h(\overline{M}_{t\wedge T_c}) \frac{rDD_{t\wedge T_c}}{c} + h^{\wedge}(\overline{M}_{t\wedge T_c}) \left(1 - \frac{rDD_{t\wedge T_c}}{c}\right),
\end{aligned}$$

where

$$h^{\wedge}(x) = \frac{1}{c} x^{1/c} \int_x^{\infty} h(y) y^{-(1+c)/c} dy.$$

compare to the ...

Russian options of Shepp and Shiryaev (1993)

crash options of Vecer (2007)

drawdown–drawup options

of Carr–Hadjiliadis–Zhang (2010)