Optimal Investment and Consumption Decision of Family with Life Insurance

Minsuk Kwak[1] [2] Yong Hyun Shin[3] U Jin Choi[4]

6th World Congress of the Bachelier Finance Society
Toronto, Canada
June 25, 2010

[1] Speaker
[2] Department of Mathematical Sciences, KAIST
[3] School of Computational Sciences, KIAS
[4] Department of Mathematical Sciences, KAIST

Table of Contents

Introduction

- Two economic agents in family: Parents and Children.
- Parents' lifetime is uncertain.
- We assume that parents represent children's father or mother with labor income, and that children have no labor income.
- While alive, parents receive deterministic labor income until $T > 0$.
- If parents die before T, the children have no income until T and they choose the optimal consumption and portfolio with remaining wealth combining the insurance benefit.
- Consider utility functions of parents and children separately.
- Maximize the weighted average of utility of parents and utility of children.
- Using the martingale method, analytic solutions for the value function and the optimal policies are derived.
- We analyze how the changes of the weight of parents' utility function and other factors, such as family's current wealth level and the fair discounted value of future labor income, affect the optimal policies and also illustrate some numerical examples.

Literature review(Selected)

- Yaari (1965)
- Richard (1975)
- Pliska and Ye (2007): studied optimal life insurance and consumption for a income earner whose lifetime is random and unbounded.
- Ye (2007): considerd optimal life insurance, consumption and portfolio choice problem under uncertain lifetime using martingale method as we used to solve our problem.
- Bayraktar and Young (2008): solved the problem of maximizing utility of consumption with a constraint on the probability of lifetime ruin, which can be interpreted as a risk measure on the whole path of the wealth process.
- Huang *et al.* (2008): investigated optimal life insurance, consumption and portfolio choice problem under uncertain lifetime with stochastic income process. They focussed on the effect of correlation between the dynamics of financial capital and human capital.

Financial market composed by two assets

Financial market

It is assumed that there are one risk-free asset and one risky asset.

- Risk-free asset:

$$dS_t^0 = rS_t^0 dt \tag{1}$$

- Risky asset:

$$dS_t^1 = \mu S_t^1 dt + \sigma S_t^1 dW_t \tag{2}$$

- W_t: a standard Brownian motion on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$
- $\{\mathcal{F}_t\}_{t=0}^T$ is the $\mathbb{P}$-augmentation of the natural filtration generated by W_t.
- r, μ, σ : constants

Definitions

Control variables

- $\pi(t)$: amount invested in the risky asset S_t^1 at time t
- $c_p(t)$: consumption rate of parents at time t
- $c_c(t)$: consumption rate of children at time t
- $I(t)$: life insurance premium rate at time t

Notations

- w_t: deterministic labor income of parents
- $\theta \triangleq \frac{\mu - r}{\sigma}$: market-price-of-risk:
- $\zeta_t \triangleq e^{-\int_0^t (\lambda_{y+s}+r)ds}$: discount process
- $Z_t \triangleq e^{-\theta W_t - \frac{1}{2}\theta^2 t}$: exponential martingale process
- $H_t \triangleq \zeta_t Z_t$: pricing kernel(state-price-density) process

Uncertain life time

Law of mortality λ_{y+t}

Let λ_{y+t} be an instantaneous force of mortality curve (hazard rate), where y is the age of the breadwinner at initial time of the model.
Then the conditional probability of survival, from age y to $y+t$, under the law of mortality defined by λ_{y+t} can be computed by

$$_t p_y \triangleq e^{-\int_0^t (\lambda_{y+s}) ds}. \tag{3}$$

Family's wealth dynamics

Life insurance benefit

- Family's insurance premium rate at time t is $I(t)$
- Receive lump sum payment $\frac{I(\tau)}{\lambda_{y+\tau}}$ at the parents' death time τ.
- X_t : family's wealth at time t until $\tau_m \triangleq \min[\tau, T]$
- Define $M(t) \triangleq X_t + \frac{I(t)}{\lambda_{y+t}}$: total legacy when the parents die at time t with wealth X_t

Family's wealth dynamics

The family's wealth dynamics X_t satisfies the following SDE:

$$dX_t = [rX_t + (\mu - r)\pi(t) - c_p(t) - c_c(t) - I(t) + w_t]dt + \sigma\pi(t)dW_t \tag{4}$$
$$= [(r + \lambda_{y+t})X_t + (\mu - r)\pi(t) - c_p(t) - c_c(t) - \lambda_{y+t}M(t) + w_t]dt + \sigma\pi(t)dW_t,$$

for $0 \leq t < \tau_m$.

Family's wealth dynamics

Equivalent martingale measure

For a given T, we define the equivalent martingale measure

$$\widetilde{\mathbb{P}}(A) \triangleq \mathbb{E}[Z_T \mathbf{1}_A], \quad \text{for} \quad A \in \mathcal{F}_T.$$

By Girsanov's theorem, $\widetilde{W}_t \triangleq W_t + \theta t,\ 0 \le t \le T$, is a standard Brownian motion under the new measure $\widetilde{\mathbb{P}}$.

Family's wealth dynamics under $\widetilde{\mathbb{P}}$

The wealth process (4) before τ_m can be rewritten as

$$\begin{aligned} dX_t &= [rX_t - c_p(t) - c_c(t) - I(t) + w_t]dt + \sigma\pi(t)d\widetilde{W}_t \\ &= [(r + \lambda_{y+t})X_t - c_p(t) - c_c(t) - \lambda_{y+t}M(t) + w_t]dt + \sigma\pi(t)d\widetilde{W}_t, \end{aligned} \tag{5}$$

for $0 \le t < \tau_m$.

Budget constraint

Budget constraint

We have the following budget constraint

$$\mathbb{E}_t\left[\int_t^T H_s c_p(s)ds + \int_t^T H_s c_c(s)ds + \int_t^T \lambda_{y+s} H_s M(s)ds + H_T X_T\right] \leq H_t(X_t + b_t), \text{ for } 0 \leq t < \tau_m, \quad (6)$$

where

$$b_t \triangleq \int_t^T w_s \frac{\zeta_s}{\zeta_t} ds^a.$$

[a] b_t is the fair discounted value of the parents' future labor income from t to τ_m.

Optimization problem of family

Expected utility at time t

Family's expected utility function $U(t, X_t; c_p, c_c, \pi, l)$ with an initial endowment X_t at time t, $t < \tau_m$:

$$U(t, X_t; c_p, c_c, \pi, l) = \mathbb{E}_t\left[\alpha_1 \int_t^{\tau_m} e^{-\delta(s-t)} u_p(c_p(s))ds + \alpha_2 \int_t^{T} e^{-\delta(s-t)} u_c(c_c(s))ds\right]. \qquad (7)$$

- $u_p(c)$: utility function of parents
- $u_c(c)$: utility function of children
- $\delta > 0$: constant subjective discount rate
- $\alpha_1 \geq 0$: constant weights of utility function of parents
- $\alpha_2 \geq 0$: constant weights of utility function of children
- α_1 and α_2 satisfies

$$\alpha_1 + \alpha_2 = 1.$$

Optimization problem of family

Power utility functions with lower bound of consumption rate

Utility function of parents $u_p(c)$:

$$u_p(c) \triangleq \frac{(c - R_p)^{1-\gamma_p}}{1 - \gamma_p} \quad (8)$$

Utility function of children $u_c(c)$:

$$u_c(c) \triangleq \frac{(c - R_c)^{1-\gamma_c}}{1 - \gamma_c}. \quad (9)$$

- $\gamma_p > 0(\gamma_p \neq 1)$: parents' coefficient of relative risk aversion
- $\gamma_c > 0(\gamma_c \neq 1)$: children's coefficient of relative risk aversion
- $R_p \geq 0$: lower bound of parents' consumption rate
- $R_c \geq 0$: lower bound of children's consumption rate

Merton's constant

Assumption 1

We define the Merton's constant $K_i,\ i = p,\ c$, and assume that it is always positive, that is,

$$K_i \triangleq r + \frac{\delta - r}{\gamma_i} + \frac{\gamma_i - 1}{2\gamma_i^2}\theta^2 > 0, \quad i = p,\ c.$$

Steps to find the value function

Step1

We first solve the optimization problem after τ_m, $t \geq \tau_m$,

Step2

and then solve the problem before τ_m, $t < \tau_m$, using martingale methods.

Steps to find the value function

Step1

We first solve the optimization problem after τ_m, $t \geq \tau_m$,

Step2

and then solve the problem before τ_m, $t < \tau_m$, using martingale methods.

Step1: Optimization problem after τ_m

Optimization problem for $\tau_m \leq t \leq T$

- If parents die before T, that is, $\tau < T$, then $w_t = 0$ for $\tau \leq t \leq T$.
- If $\tau < T$, $l(t) = 0$ for $\tau \leq t \leq T$.
- Therefore, children have only two control variables: their consumption $c_c(t)$, and investment $\pi(t)$.
- Family's expected utility function $U_c(t, X_t; c_c, \pi)$ with an initial endowment X_t at time t, $\tau_m \leq t \leq T$:

$$U_c(t, X_t; c_c, \pi) = \mathbb{E}_t \left[\int_t^T e^{-\delta(s-t)} u_c(c_c(s)) ds \right]. \tag{10}$$

- For $\tau_m \leq t \leq T$, let $\mathcal{A}_c(t, X_t)$ be the admissible class of the pair (c_c, π) at time t for which the family's expected utility function (10) is well-defined.

Optimization problem for $\tau_m \le t \le T$

Lemma 1

For $\tau_m \le t \le T$, the value function is

$$V_c(t, X_t) \triangleq \sup_{(c_c,\pi)\in\mathcal{A}_c(t,X_t)} U_c(t, X_t; c_c, \pi) = e^{\delta t}\Phi(t, X_t),$$

where

$$\Phi(t, X_t) \triangleq e^{-\delta t}\frac{g(t)^{\gamma_c}}{1-\gamma_c}\left\{X_t - \frac{R_c}{r}\left(1 - e^{-r(T-t)}\right)\right\}^{1-\gamma_c}$$

and

$$g(t) \triangleq \frac{1 - e^{-K_c(T-t)}}{K_c}.$$

Optimization problem for $\tau_m \leq t \leq T$

Lemma 1(Continued)

And the optimal policies are given by

$$c_c^*(t) = \frac{1}{g(t)} \left\{ X_t - \frac{R_c}{r} \left(1 - e^{-r(T-t)}\right) \right\} + R_c$$

and

$$\pi^*(t) = \frac{\theta}{\sigma\gamma_c} \left\{ X_t - \frac{R_c}{r} \left(1 - e^{-r(T-t)}\right) \right\}.$$

Optimization problem for $t < \tau_m$

Value function for $t < \tau_m$

The value function of the family at time $t < \tau_m$, $V(t, X_t)$, is defined as follows:

$$V(t, X_t) \triangleq \sup_{(c_p, c_c, \pi, I) \in \mathcal{A}(t, X_t)} \mathbb{E}_t \left[\int_t^{\tau_m} e^{-\delta(s-t)} \{\alpha_1 u_p(c_p(s)) + \alpha_2 u_c(c_c(s))\} ds + \alpha_2 \mathbf{1}_{\{\tau < T\}} e^{\delta t} \Phi(\tau, M(\tau)) \right] \quad (11)$$

subject to the budget constraint (6), where $\mathcal{A}(t, X_t)$ is the admissible class of the quadruple (c_p, c_c, π, I) at time $t < \tau_m$ for which the family's expected utility function (7) is well-defined.

Value function for $t < \tau_m$

Theorem 1

The *value function* $V(t, X_t)$ *is given by*

$$V(t,X_t) = Y_t^{\nu^*}\left\{X_t + b_t - R_p\int_t^T e^{-\int_t^s(\lambda_{y+u}+r)du}ds - \frac{R_c}{r}\left(1-e^{-r(T-t)}\right)\right\}$$
$$+\alpha_1^{\frac{1}{\gamma_p}}\frac{\gamma_p}{1-\gamma_p}\left(Y_t^{\nu^*}\right)^{\frac{\gamma_p-1}{\gamma_p}}\int_t^T e^{-\int_t^s(\lambda_{y+u}+K_p)du}ds + \alpha_2^{\frac{1}{\gamma_c}}\frac{\gamma_c}{1-\gamma_c}\left(Y_t^{\nu^*}\right)^{\frac{\gamma_c-1}{\gamma_c}} g(t),$$

where $Y_t^{\nu^*}$ *satisfies the following equation:*[a]

$$X_t = \alpha_1^{\frac{1}{\gamma_p}}\left(Y_t^{\nu^*}\right)^{-\frac{1}{\gamma_p}}\int_t^T e^{-\int_t^s(\lambda_{y+u}+K_p)du}ds + \alpha_2^{\frac{1}{\gamma_c}}\left(Y_t^{\nu^*}\right)^{-\frac{1}{\gamma_c}} g(t)$$
$$+ R_p\int_t^T e^{-\int_t^s(\lambda_{y+u}+r)du}ds + \frac{R_c}{r}\left(1-e^{-r(T-t)}\right) - b_t. \tag{12}$$

[a] Given t and X_t, $Y_t^{\nu^*}$ is uniquely determined by the equation (12).

Theorem 1(Continued)

For $0 \le t < \tau_m$*, the* *optimal policies* *are given by*

$$c_p^*(t) = \alpha_1^{\frac{1}{\gamma_p}} \left(Y_t^{\nu^*} \right)^{-\frac{1}{\gamma_p}} + R_p, \quad c_c^*(t) = \alpha_2^{\frac{1}{\gamma_c}} \left(Y_t^{\nu^*} \right)^{-\frac{1}{\gamma_c}} + R_c,$$

$$\pi^*(t) = \frac{\theta}{\sigma\gamma_c} \left\{ X_t + b_t - R_p \int_t^T e^{-\int_t^s (\lambda_{y+u}+r)du} ds - \frac{R_c}{r} \left(1 - e^{-r(T-t)} \right) \right\}$$
$$+ \frac{\theta}{\sigma} \frac{\gamma_c - \gamma_p}{\gamma_p \gamma_c} \alpha_1^{\frac{1}{\gamma_p}} \left(Y_t^{\nu^*} \right)^{-\frac{1}{\gamma_p}} \int_t^T e^{-\int_t^s (\lambda_{y+u}+K_p)du} ds, \tag{13}$$

and

$$\frac{I^*(t)}{\lambda_{y+t}} = M^*(t) - X_t$$
$$= b_t - R_p \int_t^T e^{-\int_t^s (\lambda_{y+u}+r)du} ds - \alpha_1^{\frac{1}{\gamma_p}} \left(Y_t^{\nu^*} \right)^{-\frac{1}{\gamma_p}} \int_t^T e^{-\int_t^s (\lambda_{y+u}+K_p)du} ds. \tag{14}$$

Properties of Optimal Policies

Proposition 1

If $\alpha_1 \in (0, 1]$, the optimal life insurance premium rate $I^*(t)$ decreases as the wealth level X_t increases. If $\alpha_1 = 0$, the optimal life insurance premium rate $I^*(t)$ is not affected by X_t.

Proposition 2

If $\alpha_1 \in [0, 1)$, the optimal life insurance premium rate $I^*(t)$ increases as the fair discounted value of future labor income b_t increases. If $\alpha_1 = 1$, the optimal life insurance premium rate $I^*(t)$ is not affected by b_t.

Properties of Optimal Policies

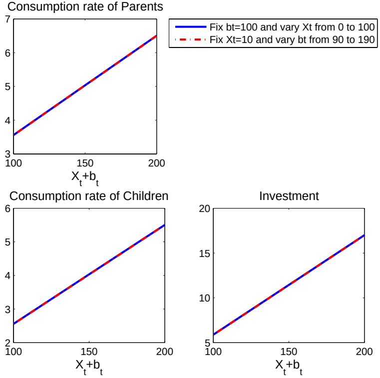

Figure 1: Relations between the optimal policies and $X_t + b_t$.($t = 0$, $R_p = 2$, $R_c = 1$, $\delta = 0.03$, $r = 0.04$, $\mu = 0.06$, $\sigma = 0.3$, $A_L = 0.005$, $B_L = 0.001125$, $T = 30$)

Remarks about Figure 1

- Solid lines represent the relations between the optimal policies and X_t, whereas dotted lines represent the relations between the optimal policies and b_t.
- We can observe that the optimal policies except the optimal life insurance premium rate $I^*(t)$ are determined by $X_t + b_t$.

Properties of Optimal Policies

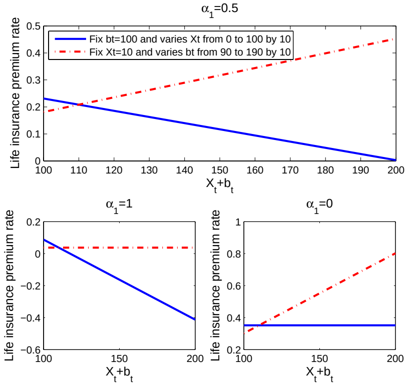

Figure 2: Relations between the optimal policies and $X_t + b_t$.($t = 0$, $R_p = 2$, $R_c = 1$, $\delta = 0.03$, $r = 0.04$, $\mu = 0.06$, $\sigma = 0.3$, $A_L = 0.005$, $B_L = 0.001125$, $T = 30$)

Remarks about Figure 2

- Unlike other optimal policies, $I^*(t)$ is not determined by $X_t + b_t$.
- It can be seen that the current wealth level X_t has a negative effect on the optimal life insurance premium rate $I^*(t)$.
- On the other hand, b_t has a positive effect on $I^*(t)$.
- Therefore, the optimal life insurance premium rate $I^*(t)$ is determined not by $X_t + b_t$, but by both X_t and b_t.

Properties of Optimal Policies

Proposition 3

The optimal life insurance premium rate $I^*(t)$ decreases as the weight of the utility function of parents α_1 increases from 0 to 1.

Proposition 4

The optimal investment $\pi(t)^*$ decreases as α_1 increases from 0 to 1 if $\gamma_p > \gamma_c$.
The optimal investment $\pi(t)^*$ increases as α_1 increases from 0 to 1 if $\gamma_p < \gamma_c$.
If $\gamma_p = \gamma_c$, $\pi(t)^*$ is not affected by α_1.

Properties of Optimal Policies

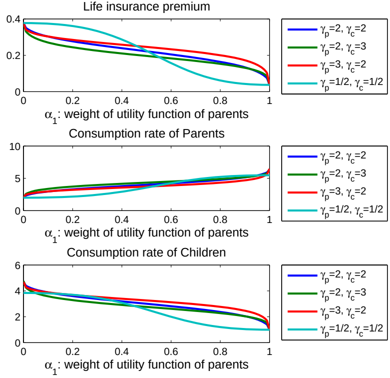

Figure 3: Relations between α_1 and the optimal policies.
($t = 0$, $X_0 = 10$, $R_p = 2$, $R_c = 1$, $\delta = 0.03$, $r = 0.04$, $\mu = 0.06$, $\sigma = 0.3$, $C_w = 5.0$, $k_w = 0.03$, $A_L = 0.005$, $B_L = 0.001125$, $T = 30$)

Remarks about Figure 3

- Figure 3 illustrate the relations between α_1 and the optimal policies for different risk aversion.
- Since α_1 is the weight of the utility function of parents, it is obvious that the optimal consumption of parents $c_p^*(t)$ increases and the optimal consumption of children $c_c^*(t)$ decreases as α_1 increases.
- The optimal life insurance premium rate $I^*(t)$ decreases as α_1 increases. This is because bequest motive is weak when α_1 is large.

Properties of Optimal Policies

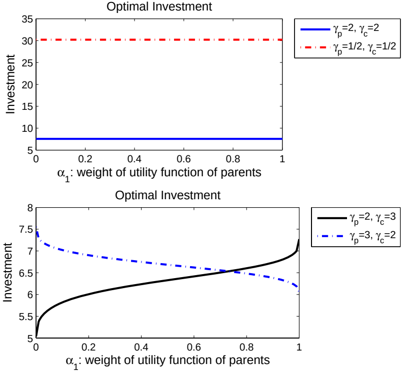

Figure 4: Relations between α_1 and the optimal policies.
($t = 0$, $X_0 = 10$, $R_p = 2$, $R_c = 1$, $\delta = 0.03$, $r = 0.04$, $\mu = 0.06$, $\sigma = 0.3$, $C_w = 5.0$, $k_w = 0.03$, $A_L = 0.005$, $B_L = 0.001125$, $T = 30$)

Remarks about Figure 4

- Unlike other optimal policies, the effect of α_1 on the optimal investment $\pi^*(t)$ depends on risk aversion.
- If $\gamma_p > \gamma_c$, $\pi^*(t)$ decreases as α_1 increases, and $\pi^*(t)$ increases as α_1 increases when $\gamma_p < \gamma_c$.
- In other words, if the risk aversion coefficients of parents and children are different, $\pi^*(t)$ increases when the relative importance of less risk averse family member's utility increases.
- If $\gamma_p = \gamma_c$, $\pi^*(t)$ is not affected by α_1.

Concluding Remarks

- We investigate an optimal portfolio, consumption and life insurance premium choice problem of family.
- Analytic solutions for the value function and the optimal policies are derived by the martingale method.
- We analyze the properties of the optimal policies, where the emphasis is placed on the role of α_1 which is the weight of parents' utility function.

Concluding Remarks

- We investigate an optimal portfolio, consumption and life insurance premium choice problem of family.
- Analytic solutions for the value function and the optimal policies are derived by the martingale method.
- We analyze the properties of the optimal policies, where the emphasis is placed on the role of α_1 which is the weight of parents' utility function.

Concluding Remarks

- We investigate an optimal portfolio, consumption and life insurance premium choice problem of family.
- Analytic solutions for the value function and the optimal policies are derived by the martingale method.
- We analyze the properties of the optimal policies, where the emphasis is placed on the role of α_1 which is the weight of parents' utility function.

Thank you!