Discrete-Time Minimum-Variance Hedging of European Contingent Claims

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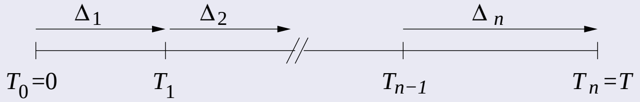
Pricing and Hedging of Financial Derivatives

- Derivative: A financial asset which derives its present value from the uncertain future value of an underlying risky asset
  - Pricing problem: To determine a fair present value of the derivative
  - Hedging problem: To determine a trading strategy to minimize the seller's risk
- Black and Scholes (1973), Merton (1973): If the underlying asset price follows a geometric Brownian motion (GBM), then
  - There exists a self-financed hedging portfolio that replicates the derivative
  - The initial investment on the portfolio gives the unique fair price of the derivative
- **Catch:** The replicating strategy requires trading in continuous time

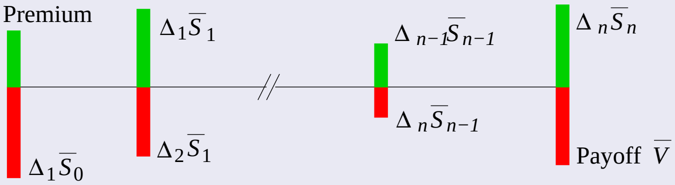
Hedging under Discrete-Time Trading

- In practice, observations and trades possible only at discrete times
- Transaction costs make frequent trading potentially expensive
- Exact replication and complete elimination of risk not possible with discrete-time trading
- Look for strategies to minimize various measures of risk
  - Föllmer and Schweizer (1989), Schäl (1994), Schweizer (1996)
- Apply strategy to a particular market model such as GBM
  - Angelini and Herzel (2009): Minimum-variance hedging for a European call option
  - **Our Objective:** To extend to general European-type derivatives including path dependent options

Discrete-Time Hedging



Discrete−time hedging



Discounted cash flows

The Minimum-Variance Hedging Problem

- **Problem:** Determine the positions $\Delta_1, \ldots, \Delta_n$ in terms of available information such that the **risk-neutral** variance of the final money position is minimized
- **Assumption:** All relevant random variables are square integrable
- **Solution:** (Föllmer and Schweizer (1989), Schäl (1994))

$$\boxed{\Delta_k^* = \frac{\operatorname{covar}(S_k, V_k | \mathcal{F}_{k-1})}{\operatorname{var}(S_k | \mathcal{F}_{k-1})}}$$

  - $\mathcal{F}_k = \sigma$-algebra generated by underlying asset prices observed upto T_k
  - $V_k =$ arbitrage-free price of ECC at T_k
- A model for the underlying asset price process is required for computing the strategy

Minimum-Variance Hedging for GBM

$$dS_t = rS_t dt + \sigma S_t dW_t$$

$$S_t = S_0 \exp\left[(r - \tfrac{1}{2}\sigma^2)t + \sigma W_t\right]$$

- Need to compute $\mathbb{E}(S_k V_k | \mathcal{F}_{k-1})$
- In case of simple claims
    - V_k is an explicit function $v(T_k, S_k)$ of S_k,
    - Given S_{k-1}, S_k is log normal

$$\mathbb{E}(S_k V_k | \mathcal{F}_{k-1}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} v\left(T_k, S_{k-1} e^x\right) S_{k-1} e^x e^{-\frac{(x-\mu)^2}{2\Lambda^2}} dx$$

- For a general path-dependent claim
    - V_k may depend on other path-dependent variables
    - The joint distributions of those variables with S_k will be needed

Preliminaries on the Wiener Space

- *Wiener space* $\Omega =$ linear vector space of continuous functions on $[0,T]$ zero at $t=0$, equipped with topology of uniform convergence
- *Wiener measure* P, the "distribution" of the Wiener process
- Coordinate process $W(t,\omega) = \omega(t)$ is a Wiener process
- **Cameron-Martin Theorem**
  - Suppose $h \in \Omega$ is absolutely continuous and $\dot{h}$ is square integrable
  - Define $\tau^h : \Omega \to \Omega$ by $\tau^h(\omega) = \omega + h$
  - Let $P^h =$ push-forward of P by τ^h, that is, $P^h(A) = P(\tau^{-h}(A))$
  - Then $\frac{\mathrm{d}P^h}{\mathrm{d}P} = \exp\left[\int_0^T \dot{h}(s)\mathrm{d}W_s - \frac{1}{2}\int_0^T \dot{h}^2(s)\mathrm{d}s\right]$

$$\mathbb{E}\left(X\frac{\mathrm{d}P^h}{\mathrm{d}P}\middle|\mathcal{F}_t\right) = \mathbb{E}^{P^h}(X|\mathcal{F}_t)\mathbb{E}\left(\frac{\mathrm{d}P^h}{\mathrm{d}P}\middle|\mathcal{F}_t\right)$$

- If $\dot{h} \equiv 0$ on $[t,T]$, then $\mathbb{E}^{P^h}(X|\mathcal{F}_t) = \mathbb{E}(X|\mathcal{F}_t)$
- If $h \equiv 0$ on $[0,t]$, then $\mathbb{E}^{P^h}(X|\mathcal{F}_t) = \mathbb{E}(X \circ \tau^h|\mathcal{F}_t)$

- To find $\mathbb{E}(V_k S_k|\mathcal{F}_{k-1})$

$$\begin{aligned} S_k &= S_{k-1}\exp\left[(r-\tfrac{1}{2}\sigma^2)(T_k-T_{k-1})+\sigma(W_k-W_{k-1})\right] \\ &= S_{k-1}e^{r\delta_k}\exp\left[\int\limits_0^T \dot{h}_k(s)\mathrm{d}W_s-\tfrac{1}{2}\int\limits_0^T \dot{h}_k^2(s)\mathrm{d}s\right]=S_{k-1}e^{r\delta_k}\frac{\mathrm{d}P^{h_k}}{\mathrm{d}P} \end{aligned}$$

$$h_k(t)\triangleq\int_0^t\sigma\chi_{[T_{k-1},T_k]}(s)\mathrm{d}s,\quad \delta_k\triangleq T_k-T_{k-1}$$

$$\mathbb{E}(V_k S_k|\mathcal{F}_{k-1})=e^{r\delta_k}e^{-r(T-T_k)}S_{k-1}\mathbb{E}(V\circ\tau^{h_k}|\mathcal{F}_{k-1})$$

$$\boxed{\Delta_k^*=\frac{e^{-r(T-T_{k-1})}}{(e^{\sigma^2\delta_k}-1)S_{k-1}}\mathbb{E}(V\circ\tau^{h_k}-V|\mathcal{F}_{k-1})}$$

Some Observations

- Numerator (denominator) $=$ conditional expectation of change in payoff (asset price) when Wiener paths are shifted by the function h_k
- Extends to time-varying (but deterministic) volatility
- If V is a functional of the stock path, then $V \circ \tau^{h_k}$ is the same functional of the modified asset-price path $\hat{S}_t \triangleq e^{\sigma h_k(t)} S_t$ satisfying

$$\mathrm{d}\hat{S}_t = (r + \sigma \dot{h}_k(t))\hat{S}_t \mathrm{d}t + \sigma \hat{S}_t \mathrm{d}W_t$$

  - Modified asset-price path follows a GBM with time-varying (but deterministic) interest rate
  - A closed-form pricing solution for the modified GBM will yield a closed-form solution for the minimum-variance hedging strategy

Example: Simple Claims

- Payoff determined solely by underlying asset price at maturity
- If $V = \phi(S_T)$, then $V \circ \tau^{h_k} = \phi(e^{\sigma^2 \delta_k} S_T)$
- ECC price at time t determined solely by S_t
- If $v(t, S_t) = e^{-r(T-t)} \mathbb{E}(V | \mathcal{F}_t)$, then $e^{-r(T-T_{k-1})} \mathbb{E}(V \circ \tau^{h_k} | \mathcal{F}_{k-1}) = v(t, e^{\sigma^2 \delta_k} S_{k-1})$

$$\boxed{\Delta_k^* = \frac{v(T_{k-1}, e^{\sigma^2 \delta_k} S_{k-1}) - v(T_{k-1}, S_{k-1})}{(e^{\sigma^2 \delta_k} - 1) S_{k-1}}}$$

Some More Observations

- If the price has a closed-form solution then the minimum variance hedging strategy is easily computable
- In contrast, the delta-neutral needs differentiability of the pricing function
- Monte-Carlo can be used in general
- Expand Δ_k^* using Taylor series in δ_k

$$\Delta_k^* = \underbrace{\frac{\partial v}{\partial x}(T_{k-1}, S_{k-1})}_{\text{Black-Scholes' } \triangle} + \underbrace{\tfrac{\sigma^2 \delta_k}{2} S_{k-1} \frac{\partial^2 v}{\partial x^2}(T_{k-1}, S_{k-1})}_{\text{Wilmott's correction}} + o(\delta_k^2)$$

- Variance approaches zero as hedging frequency increases, perfect replication achieved in the limit

Example: Asian-type Claims

- Payoff determined by underlying asset price and its continuously sampled geometric average at maturity

$$
\begin{aligned}
V &= \phi(S_T, G_T),\ G_t \stackrel{\triangle}{=} \exp\left[\frac{1}{T}\int_0^t \log S_u \mathrm{d}u\right] S_t^{(T-t)/T} \\
V \circ \tau^{h_k} &= \phi(e^{\sigma^2\delta_k} S_T, e^{\sigma^2\eta_k} G_T),\ \eta_k \stackrel{\triangle}{=} \frac{\delta_k}{T}\left[T - \frac{1}{2}(T_k + T_{k-1})\right]
\end{aligned}
$$

- ECC price at t determined by S_t and G_t
- If $v(t, S_t, G_t) = e^{-r(T-t)}\mathbb{E}(V|\mathcal{F}_t)$, then $e^{-r(T-T_{k-1})}\mathbb{E}(V \circ \tau^{h_k}|\mathcal{F}_{k-1}) = v(t, e^{\sigma^2\delta_k}S_{k-1}, e^{\sigma^2\eta_k}G_{k-1})$

$$
\boxed{\Delta_k^* = \frac{v(T_{k-1}, e^{\sigma^2\delta_k}S_{k-1}, e^{\sigma^2\eta_k}G_{k-1}) - v(T_{k-1}, S_{k-1}, G_{k-1})}{(e^{\sigma^2\delta_k} - 1)S_{k-1}}}
$$

Summary and Ongoing Work

- Minimum-variance hedging strategy for a path-dependent ECC
  - Involves pricing the ECC when Wiener paths are shifted
  - Easy to implement numerically
  - Can be expressed in terms of pricing functions
  - Closed-form expressions possible for loglinear claims
    - Simple, Asian-Geometric, Cliquet, Multi-Look Options
- Possible extensions
  - Multi-asset options such as exchange and basket options
  - Options involving random exercise or knock out
  - Stochastic volatility models
  - Hedging using a portfolio of derivative assets