Arbitrage for investment-production model in discrete time with proportional transaction costs

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Content

Content

A practical problem on Electricity market

Two challenges on Electricit markets :

- On an Electricity Spot&Future market, a producer can hedge an option within the market or by selling the produced good $\Rightarrow$ A production-investment model.
- Illiquidity and transport difficulties (of production ressources) $\Rightarrow$ Transaction Costs.

Framework : Situation where a producer can invest on a market with proportional transaction costs, or produce some assets *with a non-linear production function*.

Other example : Coal extractor hedging his production with future contracts.

Introduction to models with transaction costs

- Let $\pi := (\pi_t)_{t \leq 0} \subset L^0(\mathbb{M}^d, \mathbb{F})$ be the exchange price from one unit of i to one unit of j , such that
  - (i) $\pi_t^{ii} = 1$ (conservation of portfolio)
  - (ii) $\pi_t^{ij} > 0$ (prices are positives)
  - (iii) $\pi_t^{ij}\pi_t^{jk} \geq \pi_t^{ik}$ (direct transferts are better)
- If S is a (fictitious) price of d assets, then $\frac{1}{\pi^{ji}} \leq \frac{S^j}{S^i} \leq \pi^{ij}$.
- For a transaction cost λ_t^{ij} : $\pi^{ij} = \frac{S^j}{S^i}(1 + \lambda_t^{ij})$.
- π is also called the *bid-asked process* (e.g. Campi & Schachermayer (2006))

The geometrical formulation

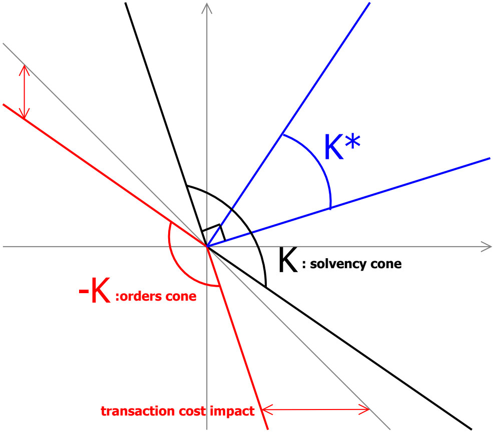
- Each exchange order (from i to j) is of the form $\lambda(e_j - \pi^{ij} e_i)$, $\lambda \in \mathbb{R}_+$.
- One can throw some asset $i \leq d$, throwing order is of the form $\lambda(-e_i)$, $\lambda \in \mathbb{R}_+$.
- Linear combinations of orders put evolutions of the portfolio in

$$-K(\omega) := \mathrm{conv}\{e_j - \pi^{ij}(\omega) e_i, -e_i \ ; \ i,j \leq d\}.$$

- A solvable position V is such that an admissible order $\xi \in -K$ can clear the position

$$V + \xi = 0 \quad \Rightarrow \quad V \in K \ .$$

A comprehensive geometrical interpretation



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Notations and the linear model

- $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{F} = (\mathcal{F}_t)_{t\in\mathbb{T}}$ and $\mathbb{T} := \{0, 1, \cdots, T\}$.
- $K = (K_t)_{t\in\mathbb{T}}$. K_t a $\mathbb{F}_t$-measurable random convex cone in $\mathbb{R}^d$.
- $(K_t^*)_{t\in\mathbb{T}}$: $K_t^* = \{y \in \mathbb{R}^d \;:\; x'y \geq 0 \;\forall x \in K_t\}$.
- $(R_t)_{t\in\mathbb{T}}$ a sequence of random maps from $\mathbb{R}^d_+$ to $\mathbb{R}^d$.
- A portfolio process is defined by

$$V_t^{\xi,\beta} = \sum_{s=0}^{t} (\xi_s - \beta_s + R_s(\beta_{s-1})\mathbf{1}_{s\geq 1})$$

with $(\xi, \beta) \in \mathcal{A}_0 := -K \times \mathbb{R}^d_+$ for all $0 \leq s \leq T$ and

$$A_t^{K,R}(T) := \left\{ \sum_{s=t}^{T} (\xi_s - \beta_s + R_s(\beta_{s-1})\mathbf{1}_{s\geq 1}), (\xi, \beta) \in \mathcal{A}_0 \right\} \text{ for } t \leq T\,.$$

A first no-arbitrage condition for the model

In Bouchard & Pham (2005), a similar model is studied. In their model, there is no arbitrage ($\mathbf{NA}^r(K, R)$) if

- we slightly decrease the transaction costs ;
- we slightly increase the production efficiency.

$\Rightarrow$ No arbitrage is possible even by a production strategy.

Problem : the condition is

- unrealistic : production is not market-constrainted ;
- unflexible : no dual condition for $\mathbf{NA}^r(K, R)$.

Objective :

- We want to allow reasonable production arbitrages.
- We want a simple dual condition.

Content

The asymptotic production function and the **NMA2** condition

Define $(L_t)_{t\in\mathbb{T}} \in L^0(\mathbb{M}^d, \mathbb{F})$ and suppose the following assumption

$$\textbf{(RL)} : \lim_{\eta\to\infty} \eta^{-1} R_t(\eta\beta) = L_t\beta.$$

Then define a linear model with attainable wealthes in

$$A_t^{K,L}(T) := \left\{ \sum_{s=t}^{T} \xi_s - \beta_s + L_s\beta_{s-1}\mathbf{1}_{s\geq t+1}, \quad (\xi,\beta) \in \mathcal{A}_0 \right\}$$

Definition (**NMA2**)

There is *no marginal arbitrage arbitrage of the second kind for high production regimes* if there exists $(c, L) \in L^\infty(\mathbb{R}^d \times \mathbb{M}^d, \mathbb{F})$ s.t.

1. $\forall\beta \in L^0(\mathbb{R}^d_+, \mathcal{F}_{t-1}),\ c_t + L_t\beta - R_t(\beta) \in L^0(K_t, \mathcal{F}_t)$,
2. There is no-arbitrage in the linear model, **NA2**^{L} holds.

The No Arbitrage condition of the second kind for a linear model

Definition (**NA2**^{L})

There is *no arbitrage of the second kind* for L if for $(\zeta, \beta) \in L^0(\mathbb{R}^d \times \mathbb{R}^d_+, \mathcal{F}_t)$, $t \leq T$

(i) $\zeta - \beta + L_{t+1}\beta \in L^0(K_{t+1}, \mathcal{F}_{t+1}) \Rightarrow \zeta \in K_t$,

(ii) $-\beta + L_{t+1}\beta \in L^0(K_{t+1}, \mathcal{F}_{t+1}) \Rightarrow \beta = 0$,

Interpretation : a "one-step" condition saying (i) only solvable positions at t lead to solvable positions at $t+1$ and (ii) the net production function $L - I$ is risky.

Extention of **NGV** : for $L \equiv 0$, (i) $\Leftrightarrow$ **NGV**.

The genuine No-Arbitrage condition, for $R \equiv 0$

Definition (**NGV**)

There is *no sure gain value* if for $\zeta \in L^0(\mathbb{R}^d, \mathcal{F}_t)$,

$$\zeta + A_t^{K,L}(T) \cap L^0(K_T, \mathcal{F}_T) \neq \emptyset \quad \Rightarrow \quad \zeta \in K_t \,.$$

Definition (**PCE**)

Prices are consistently extendable if for any $X \in L^1(\mathrm{int} K_t^*, \mathcal{F}_t)$, there exists $Z \in \mathcal{M}_t^T(\mathrm{int} K^*)$ (a martingale evolving in $\mathrm{int} K^*$) such that $Z_t = X$.

Theorem (Rasonyi (2009))

Under efficient friction ($\pi^{ij}\pi^{jk} > \pi^{ik}$), **NGV** $\Leftrightarrow$ **PCE**.

Extendable strictly consistent prices

- $\mathcal{M}_t^T(\mathrm{int}K^*)$ is the set of martingales $Z = (Z_s)_{t \le s \le T}$ evolving in the interior of K^*.
- $\mathcal{L}_t^T(\mathrm{int}\mathbb{R}_-^d)$ is the set of processes Z such that for $t \le s < T$ $\mathbb{E}\left[Z'_{s+1}(L_{s+1} - I)|\mathcal{F}_s\right] \in \mathrm{int}\mathbb{R}_-^d$.

Definition ($\mathbf{PCE}^L$)

Prices are consistently extendable for L if for any $X \in L^1(\mathrm{int}K_t^*, \mathcal{F}_t)$, there exits $Z \in \mathcal{M}_t^T(\mathrm{int}K^*) \cap \mathcal{L}_t^T(\mathrm{int}\mathbb{R}_-^d)$ such that $Z_t = X$.

Theorem (Fundamental Theorem of Asset Pricing)

if $\mathrm{int}K^* \neq \emptyset$, $\mathbf{NA2}^L \Leftrightarrow \mathbf{PCE}^L$.

Fatou-closure

(USC) : $\limsup_{\beta \to \beta_0} R_t(\beta) - R_t(\beta_0) \in -K_t$ for all $\beta_0 \in \mathbb{R}^d_+$.

Theorem

- **NA2**$^L \Rightarrow A_t^{K,L}(T)$ *is Fatou-closed.*
- **NMA2** $+$(**USC**) $\Rightarrow A_t^R(T)$ *is Fatou-closed.*

We introduce the set of wealth processes "bounded from below" :

$$A_{0b}^R(T) := \left\{ V \in A_0^R(T) \text{ s.t. } V + \kappa \in K_T \text{ for some } \kappa \in \mathbb{R}^d \right\}$$

and the support function

$$\alpha^R(Z) := \sup \left\{ \mathbb{E}\left[Z_T' V \right],\ V \in A_{0b}^R(T) \right\},\ Z \in \mathcal{M}_0^T(K^*)\,.$$

Super Replication Theorem

(Ra) : $\alpha R_t(\beta_1) + (1-\alpha) R_t(\beta_2) - R_t(\alpha\beta_1 + (1-\alpha)\beta_2) \in -K_t$
(Rb) : $R_t(\beta) \in L^\infty(\mathbb{R}^d, \mathcal{F})$ for $\beta \in L^\infty(\mathbb{R}^d_+, \mathcal{F})$.

Proposition

Assume that **NMA2** , **(USC)** *and* **(Ra), (Rb)** *hold. Let* $V \in L^0(\mathbb{R}^d, \mathcal{F})$ *be such that* $V + \kappa \in L^0(K_T, \mathcal{F})$ *for some* $\kappa \in \mathbb{R}^d$*.*
Then, the following are equivalent :
(i) $V \in A_0^R(T)$
(ii) $\mathbb{E}\left[Z_T' V\right] \leq \alpha^R(Z)$ *for all* $Z \in \mathcal{M}_0^T(\mathrm{int} K^*)$.
If **(RL)** *holds, then* (ii) *can be replaced by*
(ii') $\mathbb{E}\left[Z_T' V\right] \leq \alpha^R(Z)$ *for all* $Z \in \mathcal{M}_0^T(\mathrm{int} K^*) \cap \mathcal{L}_0^T(\mathrm{int}\mathbb{R}^d_-)$.

Portfolio optimization

Take U a $\mathbb{P}-$a.s. upper continuous, concave, random map from $\mathbb{R}^d$ to $]-\infty, 1]$, with $U(V) = -\infty$ on $\{V \notin K_T\}$. For $x_0 \in R^d$, we assume that

$$\mathcal{U}(x_0) := \left\{ V \in A_0^R(T) \ : \ \mathbb{E}\left[|U(x_0 + V)|\right] < \infty \right\} \neq \emptyset.$$

Proposition (Utility maximization)

Assume that **NMA2** , **(USC)** *and* **(Ra),(Rb)** *hold. Assume further that* $\mathcal{U}(x_0) \neq \emptyset$*. Then, there exists* $V(x_0) \in A_0^R(T)$ *such that*

$$\mathbb{E}\left[U(x_0 + V(x_0))\right] = \sup_{V \in \mathcal{U}(x_0)} \mathbb{E}\left[U(x_0 + V)\right] \ .$$

Content

Motivation and proportional transaction costs models

Model and notations

No Arbitrage of the 2^{nd} kind

No Arbitrage of the 1^{st} kind

Robust no-arbitrage

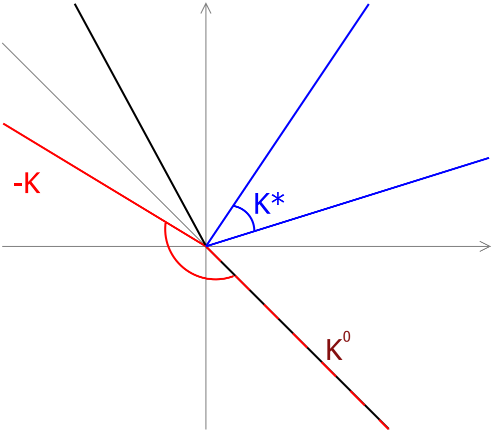
- **NMA**^{r} holds if there exists $(c, L) \in L^\infty(\mathbb{R}^d \times \mathbb{M}^d, \mathbb{F})$ s.t.
  1. $\forall \beta \in L^0(\mathbb{R}^d_+, \mathcal{F}_t),\ c_{t+1} + L_{t+1}\beta - R_{t+1}(\beta) \in L^0(K_{t+1}, \mathcal{F}_{t+1})$,
  2. **NA**$^r(K, L)$ holds.
- **NA**$^r(K, L)$ holds if (K, L) is dominated by some $(\tilde{K}, \tilde{L})$ and

$$A_t^{\tilde{K},\tilde{L}}(T) \cap L^0(\mathbb{R}^d_+, \mathcal{F}_T) = \{0\}$$

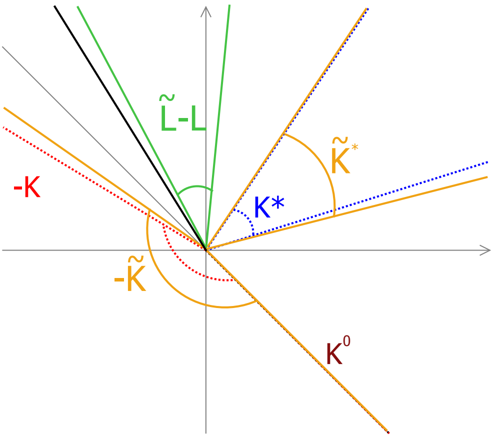
- (K, L) is dominated by $(\tilde{K}, \tilde{L})$ if for all t
  1. $K_t \backslash K_t^0 \subset \text{ri}(\tilde{K}_t)$ and $K_t \subset \tilde{K}_t$.
  2. $(\tilde{L} - L)\mathbb{R}^d_+ \subset \text{ri}(K_t)$

Here, $K_t^0 = K_t \cap -K_t$. $\text{ri}(K_t)$ stands for the relative interior of K_t.

A geometrical interpretation of $\mathbf{NA}^r(K, L)$



A geometrical interpretation of $\mathbf{NA}^r(K, L)$



Applications

Theorem (FTAP)

$\mathbf{NA}^r(K,L) \Leftrightarrow \mathcal{M}_0^T(\mathrm{int}K^*) \cap \mathcal{L}_0^T(\mathrm{int}\mathbb{R}_-^d) \neq \emptyset$

Theorem

- $\mathbf{NA}^r(K,L) \Rightarrow A_0^L(T)$ *is Fatou-closed.*
- $\mathbf{NMA}^r + (\mathbf{USC}) \Rightarrow A_0^R(T)$ *is Fatou-closed.*

Corollaries :

1. Super-hedging Theorem.
2. Existence in the Utility maximization problem.

A similar case : Bouchard & Pham (2005), the authors proved in a constraint case the super-hedging theorem, and existence of a solution in utility maximization and optimal consumption problems.

Conclusion

- Theoretical point of view :
  - Extention of **NA2** property for linear production-investment models.
  - Authorized "industrial" arbitrages for a limited regime of production for non-linear models.
  - Fatou-closure and applications to portfolio optimization.
- Practical point of view :
  - Portfolio optimization for an electricity producer knowing his production function R.
  - A setting for the formation of the competitive price of electricity.

THANK YOU FOR YOUR ATTENTION !

Theoretical interest and result

The **NGV** condition of Rasonyi (2009) :

- Extention to investment-production model.
- Fatou-closure : superhedging and utility maximization.

The $\mathbf{NA}^r(K, R)$ condition of Bouchard & Pham (2005) :

- Modification to obtain a duality result.

Yet another No Arbitrage condition for markets with propotional transaction costs *in discrete time*...

Inspiration

- B. Bouchard and H. Pham. Optimal consumption in discrete time financial models with industrial investment opportunities and non-linear returns. 2005.
- Y. Kabanov and M. Kijima. A consumption-investment problem with production possibilities. 2006.
- M. Rásonyi. Arbitrage under transaction costs revisited. 2009.
- Y. Kabanov, M. Rásonyi and C. Stricker. On the closedness of sums of convex cones in L^0 and the robust no-arbitrage property. 2003.

Submited : No marginal arbitrage of the second kind for high production regimes in discrete time production-investment models with proportional transaction costs.