Conditional Certainty Equivalent

Marco Frittelli and Marco Maggis

University of Milan

Bachelier Finance Society World Congress, Hilton Hotel, Toronto, June 25, 2010

Stochastic Dynamic Utilities

We fix a non-atomic filtered probability space

$$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$$

and suppose that the filtration is right continuous.

Stochastic Dynamic Utilities

Definition

A stochastic dynamic utility (SDU)

$$u : \mathbb{R} \times [0, \infty) \times \Omega \to \mathbb{R} \cup \{-\infty\}$$

satisfies the following conditions: for any $t \in [0, +\infty)$ there exists $A_t \in \mathcal{F}_t$ such that $\mathbb{P}(A_t) = 1$ and

Stochastic Dynamic Utilities

Definition

A stochastic dynamic utility (SDU)

$$u : \mathbb{R} \times [0, \infty) \times \Omega \to \mathbb{R} \cup \{-\infty\}$$

satisfies the following conditions: for any $t \in [0, +\infty)$ there exists $A_t \in \mathcal{F}_t$ such that $\mathbb{P}(A_t) = 1$ and

(a) the effective domain, $\mathcal{D}(t) := \{x \in \mathbb{R} : u(x, t, \omega) > -\infty\}$ and the range $\mathcal{R}(t) := \{u(x, t, \omega) \mid x \in \mathcal{D}(t)\}$ do not depend on $\omega \in A_t$; moreover $0 \in int\mathcal{D}(t)$, $E[u(0, t)] < +\infty$ and $\mathcal{R}(t) \subseteq \mathcal{R}(s)$;

Stochastic Dynamic Utilities

Definition

A stochastic dynamic utility (SDU)

$$u : \mathbb{R} \times [0, \infty) \times \Omega \to \mathbb{R} \cup \{-\infty\}$$

satisfies the following conditions: for any $t \in [0, +\infty)$ there exists $A_t \in \mathcal{F}_t$ such that $\mathbb{P}(A_t) = 1$ and

(a) the effective domain, $\mathcal{D}(t) := \{x \in \mathbb{R} : u(x, t, \omega) > -\infty\}$ and the range $\mathcal{R}(t) := \{u(x, t, \omega) \mid x \in \mathcal{D}(t)\}$ do not depend on $\omega \in A_t$; moreover $0 \in int\mathcal{D}(t)$, $E[u(0, t)] < +\infty$ and $\mathcal{R}(t) \subseteq \mathcal{R}(s)$;

(b) for all $\omega \in A_t$ and $t \in [0, +\infty)$ the function $x \to u(x, t, \omega)$ is strictly increasing on $\mathcal{D}(t)$ and increasing, concave and upper semicontinuous on $\mathbb{R}$.

Stochastic Dynamic Utilities

Definition

A stochastic dynamic utility (SDU)

$$u : \mathbb{R}\times[0,\infty) \times \Omega \to \mathbb{R} \cup \{-\infty\}$$

satisfies the following conditions: for any $t \in [0, +\infty)$ there exists $A_t \in \mathcal{F}_t$ such that $\mathbb{P}(A_t) = 1$ and

(a) the effective domain, $\mathcal{D}(t) := \{x \in \mathbb{R} : u(x, t, \omega) > -\infty\}$ and the range $\mathcal{R}(t) := \{u(x, t, \omega) \mid x \in \mathcal{D}(t)\}$ do not depend on $\omega \in A_t$; moreover $0 \in int\mathcal{D}(t)$, $E[u(0, t)] < +\infty$ and $\mathcal{R}(t) \subseteq \mathcal{R}(s)$;

(b) for all $\omega \in A_t$ and $t \in [0, +\infty)$ the function $x \to u(x, t, \omega)$ is strictly increasing on $\mathcal{D}(t)$ and increasing, concave and upper semicontinuous on $\mathbb{R}$.

(c) $\omega \to u(x, t, \cdot)$ is $\mathcal{F}_t-$measurable for all $(x, t) \in \mathcal{D}(t)\times[0, +\infty)$

Stochastic Dynamic Utilities

Occasionally we may assume that

Decreasing in time

(d) *For any fixed* $x \in \mathcal{D}(t)$, $u(x, t, \cdot) \leq u(x, s, \cdot)$ *for every* $s \leq t$.

Stochastic Dynamic Utilities

Occasionally we may assume that

Decreasing in time

(d) *For any fixed* $x \in \mathcal{D}(t)$, $u(x,t,\cdot) \leq u(x,s,\cdot)$ *for every* $s \leq t$.

We introduce the following useful

Notation:

$$\mathcal{U}(t) = \{X \in L^0(\Omega, \mathcal{F}_t, \mathbb{P}) \,|\, u(X,t) \in L^1(\Omega, \mathcal{F}, \mathbb{P})\}.$$

Stochastic Dynamic Utilities

Occasionally we may assume that

Decreasing in time

(d) *For any fixed* $x \in \mathcal{D}(t)$, $u(x,t,\cdot) \leq u(x,s,\cdot)$ *for every* $s \leq t$.

We introduce the following useful

Notation:

$$\mathcal{U}(t) = \{X \in L^0(\Omega, \mathcal{F}_t, \mathbb{P}) \,|\, u(X,t) \in L^1(\Omega, \mathcal{F}, \mathbb{P})\}.$$

Related literature:

- Series of papers by Musiela and Zariphopoulou (2006,2008,...);
- Henderson and Hobson (2007);
- Berrier, Rogers and Theranchi (2007);
- El Karoui and Mrad (2010);
- Schweizer and Choulli (2010);
- probably many other...

Conditional Certainty Equivalent

Definition

Let u be a SDU and X be a random variable in $\mathcal{U}(t)$. For each $s \in [0, t]$, the backward Conditional Certainty Equivalent $C_{s,t}(X)$ of X is the random variable in $\mathcal{U}(s)$ solution of the equation:

$$u(C_{s,t}(X), s) = E\left[u(X,t)|\mathcal{F}_s\right].$$

Thus the CCE defines the valuation operator

$$C_{s,t} : \mathcal{U}(t) \rightarrow \mathcal{U}(s),\ \ C_{s,t}(X) = u^{-1}\left(E\left[u(X,t)|\mathcal{F}_s\right], s\right).$$

Conditional Certainty Equivalent

Definition

Let u be a SDU and X be a random variable in $\mathcal{U}(t)$. For each $s \in [0, t]$, the backward Conditional Certainty Equivalent $C_{s,t}(X)$ of X is the random variable in $\mathcal{U}(s)$ solution of the equation:

$$u(C_{s,t}(X), s) = E\left[u(X, t)|\mathcal{F}_s\right].$$

Thus the CCE defines the valuation operator

$$C_{s,t} : \mathcal{U}(t) \to \mathcal{U}(s), \ \ C_{s,t}(X) = u^{-1}\left(E\left[u(X, t)|\mathcal{F}_s\right], s\right).$$

This definition is the natural generalization to the dynamic and stochastic environment of the classical definition of the certainty equivalent, as given in Pratt 1964.

Equivalent definition of the CCE

Definition (Conditional Certainty Equivalent process)

Let u be a SDU and X be a random variable in $\mathcal{U}(t)$. The backward conditional certainty equivalent of X is the only process $\{Y_s\}_{0\leq s\leq t}$ such that $Y_t \equiv X$ and the process $\{u(Y_s, s)\}_{0\leq s\leq t}$ is a martingale.

Equivalent definition of the CCE

Definition (Conditional Certainty Equivalent process)

Let u be a SDU and X be a random variable in $\mathcal{U}(t)$. The backward conditional certainty equivalent of X is the only process $\{Y_s\}_{0\leq s\leq t}$ such that $Y_t \equiv X$ and the process $\{u(Y_s, s)\}_{0\leq s\leq t}$ is a martingale.

- This definition could be compared to the definition of non linear evaluation based on g-expectation, as provided by Peng.
- Even if u is concave the CCE is not a concave functional, but it is conditionally quasiconcave

Time Consistency

Proposition

Let u be a SDU, $0 \leq s \leq v \leq t < \infty$ *and* $X, Y \in \mathcal{U}(t)$.

(i) $C_{s,t}(X) = C_{s,v}(C_{v,t}(X))$.

(ii) $C_{t,t}(X) = X$.

(iii) If $C_{v,t}(X) \leq C_{v,t}(Y)$ *then for all* $0 \leq s \leq v$ *we have:* $C_{s,t}(X) \leq C_{s,t}(Y)$. *Therefore,* $X \leq Y$ *implies that for all* $0 \leq s \leq t$ *we have:* $C_{s,t}(X) \leq C_{s,t}(Y)$. *The same holds if the inequalities are replaced by equalities.*

Regularity and Quasiconcavity

Proposition

Let u be a SDU, $0 \leq s \leq v \leq t < \infty$ *and* $X, Y \in \mathcal{U}(t)$.

(iv) Regularity: for every $A \in \mathcal{F}_s$ *we have*

$$C_{s,t}(X\mathbf{1}_A + Y\mathbf{1}_{A^C}) = C_{s,t}(X)\mathbf{1}_A + C_{s,t}(Y)\mathbf{1}_{A^C}$$

and then $C_{s,t}(X)\mathbf{1}_A = C_{s,t}(X\mathbf{1}_A)\mathbf{1}_A$.

(v) Quasiconcavity: the upper level set $\{X \in \mathcal{U}_t \mid C_{s,t}(X) \geq Z\}$ *is conditionally convex for every* $Z \in L^0_{\mathcal{F}_s}$.

Consequence of Jensen Inequality

Proposition

Let u be a SDU, $0 \leq s \leq v \leq t < \infty$ and $X, Y \in \mathcal{U}(t)$.

(vi) Suppose u satisfies (d) and for every $t \in [0, +\infty)$, $u(x,t)$ is integrable for every $x \in \mathcal{D}(t)$. Then

- $C_{s,t}(X) \leq E\left[C_{v,t}(X) | \mathcal{F}_s\right]$ *and* $E\left[C_{s,t}(X)\right] \leq E\left[C_{v,t}(X)\right]$;
- *moreover* $C_{s,t}(X) \leq E[X|\mathcal{F}_s]$ *and therefore* $E\left[C_{s,t}(X)\right] \leq E[X]$.

Example (Exponential SDU)

Let us consider $u : \mathbb{R}\times[0,\infty) \times \Omega \to \mathbb{R}$ defined by

$$u(x,t,\omega) = 1 - e^{-\alpha_t(\omega)x + A_t(\omega)}$$

where $\alpha_t > 0$ and A_t are stochastic processes.

$$C_{s,t}(X) = -\frac{1}{\alpha_s} \ln \left\{ \mathbb{E}[e^{-\alpha_t X + A_t} | \mathcal{F}_s] \right\} + \frac{A_s}{\alpha_s}.$$

If $\alpha_t(\omega) \equiv \alpha \in \mathbb{R}$ and $A_t \equiv 0$ then

$$C_{0,t}(X) = -\frac{1}{\alpha} \ln \left\{ \mathbb{E}[e^{-\alpha X}] \right\}$$

$$C_{s,t}(X) = -\frac{1}{\alpha} \ln \left\{ \mathbb{E}[e^{-\alpha X} | \mathcal{F}_s] \right\}$$

i.e. $C_{0,t}(X) = -\rho_u(X)$ where ρ_u is the risk measure induced by the exponential utility. By introducing a time dependence in the risk aversion coefficient one looses the monetary property.

Example (Exponential SDU)

Cash super-additive property:

$$C_{s,t}(X + c) \geq C_{s,t}(X) + c,\ c \in \mathbb{R}_+.$$

When the risk aversion coefficient is purely stochastic we have no chance that $C_{s,t}$ has any monetary or cash super-additive property.

Example (Exponential SDU)

Cash super-additive property:

$$C_{s,t}(X+c) \geq C_{s,t}(X) + c, \ c \in \mathbb{R}_+.$$

When the risk aversion coefficient is purely stochastic we have no chance that $C_{s,t}$ has any monetary or cash super-additive property.

Proposition

The functional $C_{s,t}(X) = -\frac{1}{\alpha_s} \ln \left\{ \mathbb{E}[e^{-\alpha_t X + A_t} | \mathcal{F}_s] \right\} + \frac{A_s}{\alpha_s}$ *is decreasing and concave.*

If the process $\{\alpha_t\}_{t \geq 0}$ *is almost surely increasing then the (CCE) is cash super-additive*

Example (Exponential SDU)

Cash super-additive property:

$$C_{s,t}(X+c) \geq C_{s,t}(X) + c, \; c \in \mathbb{R}_+.$$

When the risk aversion coefficient is purely stochastic we have no chance that $C_{s,t}$ has any monetary or cash super-additive property.

Proposition

The functional $C_{s,t}(X) = -\frac{1}{\alpha_s} \ln \left\{ \mathbb{E}[e^{-\alpha_t X + A_t} | \mathcal{F}_s] \right\} + \frac{A_s}{\alpha_s}$ *is decreasing and concave.*
If the process $\{\alpha_t\}_{t \geq 0}$ *is almost surely increasing then the (CCE) is cash super-additive*

The definition of CCE is not a priori directly linked to the existence of a market, as for the theory of forward utilities (see Musiela Zariphopoulou)

Selection of the right spaces

In literature the generalization of Orlicz spaces to the case of stochastic (not time dependent) functions are known as *Musielak – Orlicz* spaces (Musielak, *Orlicz Spaces and Modular Spaces*).
Let $u(x,t,\omega)$ be a SDU satisfying (int) condition. The **dynamic version** of Musielak-Orlicz space is given by:

$$L^{\hat{u}_t}(\mathcal{F}_t) = \left\{ X \in L^0(\mathcal{F}_t) \,\middle|\, \exists\, \lambda > 0 : \int_\Omega \hat{u}(\lambda X(\omega), t, \omega)\mathbb{P}(d\omega) < \infty \right\}$$

$$M^{\hat{u}_t}(\mathcal{F}_t) = \left\{ X \in L^0(\mathcal{F}_t) \,\middle|\, \int_\Omega \hat{u}(\lambda X(\omega), t, \omega)\mathbb{P}(d\omega) < \infty \,\forall\, \lambda > 0 \right\}$$

where $\hat{u}(x,t,\omega) = u(0,t,\omega) - u(-|x|,t,\omega)$.

Selection of the right spaces

We endow these spaces with the Luxemburg norm

$$N_{\hat{u}_t}(X) = \inf \left\{ c > 0 \,\middle|\, \int_\Omega \hat{u}\left(\frac{X(\omega)}{c}, t, \omega\right) \mathbb{P}(d\omega) \leq 1 \right\}$$

Selection of the right spaces

We endow these spaces with the Luxemburg norm

$$N_{\hat{u}_t}(X) = \inf \left\{ c > 0 \,\middle|\, \int_\Omega \hat{u}\left(\frac{X(\omega)}{c}, t, \omega\right) \mathbb{P}(d\omega) \leq 1 \right\}$$

and consider the following

Condition:

$$\int_\Omega \hat{u}(x,t,\omega)\mathbb{P}(d\omega) < \infty \qquad \textit{for every } x \in \mathcal{D}(t) \tag{int}$$

Selection of the right spaces

In general:

$$\overline{L^\infty(\mathcal{F}_t)}^{N_{\hat{u}_t}} = M^{\hat{u}_t}(\mathcal{F}_t) \subseteq L^{\hat{u}_t}(\mathcal{F}_t)$$

and if the condition (Δ_2) is satisfied then

$$M^{\hat{u}_t}(\mathcal{F}_t) = L^{\hat{u}_t}(\mathcal{F}_t)$$

Selection of the right spaces

In general:

$$\overline{L^{\infty}(\mathcal{F}_t)}^{N_{\hat{u}_t}} = M^{\hat{u}_t}(\mathcal{F}_t) \subseteq L^{\hat{u}_t}(\mathcal{F}_t)$$

and if the condition (Δ_2) is satisfied then

$$M^{\hat{u}_t}(\mathcal{F}_t) = L^{\hat{u}_t}(\mathcal{F}_t)$$

Condition:

There exists $K, x_0 \in \mathbb{R}$ and $h \in L^1$ such that

$$\Psi(2x,\cdot) \leq K\Psi(x,\cdot) + h(\cdot) \quad \textit{for all } x > x_0,\ \mathbb{P} - a.s. \qquad (\Delta_2)$$

Example (The CCE is well defined)

1) Consider an exponential dynamic utility:

$$u(x, t, \omega) = 1 - e^{-\alpha_t(\omega)x + A_t(\omega)}$$

Assume that:
$E[e^{\alpha_t |x| + A_t}] < \infty \quad \forall x \in \mathbb{R}$ and A_t belongs to $L^\infty(\mathcal{F}_t)$,

Example (The CCE is well defined)

1) Consider an exponential dynamic utility:

$$u(x,t,\omega) = 1 - e^{-\alpha_t(\omega)x + A_t(\omega)}$$

Assume that:
$E[e^{\alpha_t|x|+A_t}] < \infty \quad \forall x \in \mathbb{R}$ and A_t belongs to $L^\infty(\mathcal{F}_t)$,

Proposition

If $X \in M^{\widehat{u}_t}$ *then* $C_{s,t}(X) \in M^{\widehat{u}_s}$ *i.e.*

$$\begin{array}{rccc} C_{s,t}: & M^{\widehat{u}_t} & \longrightarrow & M^{\widehat{u}_s} \\ & X & \longmapsto & -\frac{1}{\alpha_s}\ln\left\{E[e^{-\alpha_t X + A_t}|\mathcal{F}_s]\right\} + \frac{A_s}{\alpha_s} \end{array}$$

Example (The CCE is well defined)

2)Consider a random power utility

$$u(x,t,\omega) = -\gamma_t(\omega)|x|^{p_t(\omega)}\mathbf{1}_{(-\infty,0)}$$

where γ_t, p_t are adapted stochastic processes satisfying $\gamma_t > 0$ and $p_t > 1$. In this case

$$C_{s,t}(X) = -\frac{1}{\gamma_s}\left(E[\gamma_t(X^-)^{p_t}|\mathcal{F}_s]\right)^{\frac{1}{p_s}} + K\mathbf{1}_{G^C}$$

where $K \in L^0_{\mathcal{F}_s}$, $K > 0$ and $G := \{E[\gamma_t|X|^{p_t}\mathbf{1}_{\{X<0\}}|\mathcal{F}_s] > 0\}$. If in particular $K \in M^{\hat{u}_s}$ then

$$C_{s,t} : M^{\widehat{u}_t} \longrightarrow M^{\widehat{u}_s}.$$

Example (The CCE is well defined)

3)Let $V : \mathbb{R} \to \mathbb{R}$ a concave, strictly increasing function and $\{\alpha_t\}_{t\geq 0}$ an adapted stochastic process such that for every $t \geq 0$, $\alpha_t > 0$. Then $u(x, t, \omega) = V(\alpha_t(\omega)x)$ is a SDU and

$$C_{s,t}(X) = \frac{1}{\alpha_s} V^{-1}\left(E[V(\alpha_t X) \mid \mathcal{F}_s]\right)$$

Proposition

Let $\Theta_t = \{X \in L^{\widehat{u}_t} \mid E[u(-X^-, t)] > -\infty\} \supseteq M^{\widehat{u}_t}$. *Then*

$$C_{s,t} : \Theta_t \to \Theta_s$$

Moreover if $\widehat{u}(x, s)$ *satisfies the* (Δ_2) *condition, then*

$$C_{s,t} : M^{\widehat{u}_t} \to M^{\widehat{u}_s}.$$

A good domain for the CCE

A general evidence is that

$$M^{\widehat{u}_t} \subseteq \mathcal{U}(t)$$

but

$$L^{\widehat{u}_t} \nsubseteq \mathcal{U}(t)$$

Anyway we can define the $C_{s,t}$ on the whole space $L^{\widehat{u}_t}$ using an extended version of the conditional expectation

$$E[u(X,t) \mid \mathcal{F}_s] := E[u(X,t)^+ \mid \mathcal{F}_s] - \lim_n E[u(X,t)^- \wedge n \mid \mathcal{F}_s]$$

provided that a technical assumption is satisfied.

Assumption for the dual representation

a) Rockafellar 1968: there exists $X^* \in (L^{\widehat{u}_t})^*$ s.t.

$$E[f^*(X^*, t)] < +\infty,$$

where $f^*(x, t, \omega) = \sup_{y \in \mathbb{R}} \{xy + u(y, t, \omega)\}$.

Assumption for the dual representation

b) For every fixed t, $\widehat{u}_t$ belongs to one of these three possible classes:

Assumption for the dual representation

b) For every fixed t, $\widehat{u}_t$ belongs to one of these three possible classes:

1. $\widehat{u}_t(\cdot,\omega)$ is (int) and discontinuous, i.e. $\mathcal{D}(t) \subsetneq \mathbb{R}$. In this case, $L^{\widehat{u}_t} = L^{\infty}$

Assumption for the dual representation

b) For every fixed t, $\widehat{u}_t$ belongs to one of these three possible classes:

1. $\widehat{u}_t(\cdot, \omega)$ is (int) and discontinuous, i.e. $\mathcal{D}(t) \subsetneq \mathbb{R}$. In this case, $L^{\widehat{u}_t} = L^\infty$
2. $\widehat{u}_t(\cdot, \omega)$ is continuous, $\widehat{u}_t$ and $(\widehat{u}_t)^*$ are (int) and satisfy:

$$\frac{\widehat{u}_t(x, \omega)}{x} \to +\infty \text{ , as } x \to \infty.$$

Assumption for the dual representation

b) For every fixed t, $\widehat{u}_t$ belongs to one of these three possible classes:

1. $\widehat{u}_t(\cdot, \omega)$ is (int) and discontinuous, i.e. $\mathcal{D}(t) \subsetneqq \mathbb{R}$. In this case, $L^{\widehat{u}_t} = L^\infty$
2. $\widehat{u}_t(\cdot, \omega)$ is continuous, $\widehat{u}_t$ and $(\widehat{u}_t)^*$ are (int) and satisfy:
$$\frac{\widehat{u}_t(x, \omega)}{x} \to +\infty \text{ , as } x \to \infty.$$
3. $\widehat{u}_t(\cdot, \omega)$ is continuous and
$$0 < ess \inf_{\omega \in \Omega} \lim_{x \to \infty} \frac{\widehat{u}_t(x, \omega)}{x} \leq ess \sup_{\omega \in \Omega} \lim_{x \to \infty} \frac{\widehat{u}_t(x, \omega)}{x} < +\infty$$
It follows that $L^{\widehat{u}_t} = L^1$.

The dual representation of the CCE

Theorem

For every $X \in L^{\widehat{u}_t}$

$$C_{s,t}(X) = \inf_{\mathbb{Q} \in \mathcal{P}_{\mathcal{F}_t}} G(E_{\mathbb{Q}}[X|\mathcal{F}_s], \mathbb{Q})$$

where for every $Y \in L^0_{\mathcal{F}_s}$

$$G(Y, \mathbb{Q}) = \sup_{\xi \in L^{\widehat{u}_t}} \left\{ C_{s,t}(\xi) \mid E_{\mathbb{Q}}[\xi|\mathcal{F}_s] =_{\mathbb{Q}} Y \right\}.$$

and

$$\mathcal{P}_{\mathcal{F}_t} = \left\{ \mathbb{Q} << \mathbb{P} \mid \mathbb{Q} \text{ probability and } \frac{d\mathbb{Q}}{d\mathbb{P}} \in (L^{\widehat{u}_t^*}) \right\}$$

Moreover if $X \in M^{\widehat{u}_t}$ *then the essential infimum is actually a minimum.*

THANK YOU FOR YOUR ATTENTION!!!

ANY QUESTION???