Inflation Linked Bonds: An incentive for lower inflation?

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June 23, 2010
The Basic Economic Model

Log-real output $y_t$ is given by

$$y_t = y_t^N + \tilde{a}(\pi_t - \pi_t^e),$$

where $y_t^N$ denotes the so called natural rate of output.
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In a first order approximation the central bank’s accumulated gains (in absolute real terms) over the time interval $[t; T]$ following the policy $\pi_t$ is given by

$$Y_t = Y_t^N \int_t^T \tilde{a}(\pi_s - \pi_s^e) ds.$$
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Social cost arising from inflation is assumed to be quadratic in $\pi_t$. This is weight up against the output benefit via the parameter $\tilde{\lambda}$. 

Further we assume $d\pi_e^t = \gamma (\pi_t - \pi_e^t) dt$ and $\pi_t = u^t + \sigma \dot{W}_t$. And hence one gets $d\pi_e^t = \gamma (u^t - \pi_e^t) dt + \gamma \sigma dW_t$.
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Social cost arising from inflation is assumed to be quadratic in $\pi_t$. This is weighed up against the output benefit via the parameter $\tilde{\lambda}$. Hence the bank’s instantaneous benefit function reads

$$a(\pi_t - \pi_t^e) - \frac{\tilde{\lambda}}{2} \pi_t^2 = a \left( \pi_t - \pi_t^e - \frac{\lambda}{2} \pi_t^2 \right),$$

where $a := \tilde{a} Y_t^N$ and $\lambda := \frac{\tilde{\lambda}}{a}$. 
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$$d\pi^e_t = \gamma(u_t - \pi^e_t)dt + \gamma \sigma dW_t.$$

and

$$dP_t = P_t \pi_t dt = P_t(u_t dt + \sigma dW_t).$$
Definition

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Hence the liability for the bank at maturity is given by

$$-N \left( \frac{P_T - P_s}{P_s} \right) \cdot \left( \frac{P_T}{P_s} \right)^{-1} = -N \frac{P_T - P_s}{P_T} = -N \left( 1 - \frac{P_s}{P_T} \right).$$
Using the instantaneous benefit and the approximation $1 - x \approx \log(x^{-1})$ for the terminal obligation we see the central bank needs to optimize

$$V(t, \pi^e, P, N) := \max_{u_v} \mathbb{E} \left( a \int_t^T e^{-r(v-t)} \left( u_v - \pi^e_v - \frac{\lambda}{2} u_v^2 \right) dv \right)$$

$$- e^{-r(T-t)} N \log \left( \frac{P_T}{P_s} \right) \bigg| \pi^e_t = \pi^e, P_t = P$$
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subject to

$$d\pi_t^e = \gamma (\pi_t - \pi_t^e) dt \quad ; \quad \pi_t = u_t + \sigma \dot{W}_t.$$
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This can be solved as

\[ V(t, \pi^e, P, N) = -N \log \left( \frac{P}{P_s} \right) e^{-r(T-t)} + A_t \pi^e + C_t + D_t(N) \]
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with

\[ A_t = \frac{a}{\gamma + r} \left( e^{-(\gamma+r)(T-t)} - 1 \right) \]
\[ C_t = \frac{a}{\lambda(\gamma + r)^2} \left[ \frac{r}{2} \left( 1 + e^{-r(T-t)} \right) - re^{-(\gamma+r)(T-t)} \right. \]
\[ + \frac{\gamma^2}{2(2\gamma + r)} \left( e^{-r(T-t)} - e^{-2(\gamma+r)(T-t)} \right) \]
\[ D_t(N) = e^{-r(T-t)} \left[ \frac{N^2}{2a\lambda r} \left( 1 - e^{-r(T-t)} \right) + \frac{N\sigma^2}{2} (T - t) \right. \]
\[ - \frac{N}{\lambda(\gamma + r)^2} \left( r(\gamma + r)(T - t) + \gamma \left( 1 - e^{-(\gamma+r)(T-t)} \right) \right) \]
So we get

\[
\pi_t^*(N) = \frac{1}{\lambda(\gamma + r)} + \left( \gamma e^{-(\gamma+r)(T-t)} + r \right) - \frac{Ne^{-r(T-t)}}{a\lambda} + \sigma \dot{W}_t,
\]

\[
\pi^e_t(N) = \pi_s e^{-\gamma(t-s)} + \frac{\gamma^2 e^{-(\gamma+r)T} e^{-\gamma(t-s)}}{\lambda(\gamma + r)(2\gamma + r)} \left( e^{(2\gamma+r)t} - e^{(2\gamma+r)s} \right)
+ \frac{r}{\lambda(\gamma + r)} \left( e^{\gamma t} - e^{\gamma s} \right)
- \frac{N\gamma e^{-rT} e^{\gamma(t-s)}}{a\lambda(r - \gamma)} \left( e^{(r-\gamma)t} - e^{(r-\gamma)s} \right) + \gamma \sigma e^{-\gamma(t-s)} \int_s^t e^{\gamma \nu} dW_\nu.
\]
Implications for Monetary Policy

So we get

\[
\pi_t^*(N) = \frac{1}{\lambda(\gamma + r)} + \left(\gamma e^{-(\gamma + r)(T-t)} + r\right) - \frac{Ne^{-r(T-t)}}{a\lambda} + \sigma \dot{W}_t,
\]

\[
\pi_t^{e*}(N) = \pi_t^e e^{-\gamma(t-s)} + \frac{\gamma^2 e^{-(\gamma + r)T} e^{-\gamma(t-s)}}{\lambda(\gamma + r)(2\gamma + r)} \left(e^{(2\gamma + r)t} - e^{(2\gamma + r)s}\right)
\]

\[
+ \frac{r}{\lambda(\gamma + r)} (e^{\gamma t} - e^{\gamma s})
\]

\[
- \frac{N e^{-rT} e^{\gamma(t-s)}}{a\lambda(r - \gamma)} \left(e^{(r-\gamma)t} - e^{(r-\gamma)s}\right) + \gamma \sigma e^{-\gamma(t-s)} \int_s^t e^{\gamma \nu} dW_\nu.
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Expected inflation turns negative if \( N \) is greater than

\[
\frac{ar}{(1 - e^{-r(T-s)})(\gamma + r)^2} \left(r(\gamma + r)(T - s) + \gamma(1 - e^{-(\gamma + r)(T-s)})\right).
\]
Pricing

By utility indifference pricing we get

\[ p_s(N) = e^{-r_i(T-s)} - \left[ P_s \left( \frac{D_s(N)}{N} \right) \right] \]

for normal bond and inflation compensation

for the price set by the central bank.
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for the price set by the central bank. On the demand side we assume a Black-Scholes type financial market:

- There is a nominal bond and interest rate \( r_i \).
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- There is a nominal bond and interest rate \( r_i \).
- There is a Stock with volatility \( \tilde{\sigma} \) and drift \( \mu \).
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- There is a nominal bond and interest rate \( r_i \).
- There is a Stock with volatility \( \tilde{\sigma} \) and drift \( \mu \).
- The price level is not tradeable.
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By utility indifference pricing we get

\[ p_s(N) = e^{-r_i(T-s)} - \underbrace{P_s\left(\frac{D_s(N)}{N}\right)}_{\text{normal bond}} - \underbrace{P_s\left(\frac{D_s(N)}{N}\right)}_{\text{inflation compensation}} \]

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- There is a nominal bond and interest rate \( r_i \).
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Therefore the market price of risk is \( \rho = \frac{mu-r_i}{\tilde{\sigma}} \) and under the risk free measure we have \( \tilde{E}_s(P_T) = e^{-\rho(T-s)}E(P_T) \).
By utility indifference pricing we get

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for the price set by the central bank. On the demand side we assume a Black-Scholes type financial market:

- There is a nominal bond and interest rate $r_i$.
- There is a Stock with volatility $\tilde{\sigma}$ and drift $\mu$.
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Therefore the market price of risk is $\rho = \frac{mu-r_i}{\tilde{\sigma}}$ and under the risk free measure we have $\tilde{E}_s(P_T) = e^{-\rho(T-s)}E(P_T)$. Hence the arbitrage free price is given by

$$\tilde{p}_s(N) = e^{-(r_i+\sigma\rho)(T-s)}E_s(P^*_T(N)) = e^{-r_i(T-s)}P_s e^{\int_s^T (u^*_\nu(N)-\frac{1}{2}\sigma^2)d\nu + \int_s^T \sigma d\tilde{W}_\nu}$$
Some simulation

Figure: There is excess demand for ILB’s whenever the Bank chooses $N \leq 4.067 \times 10^8$. Supply meets demand when $N = 4.067 \times 10^8$. 
Some simulation

Figure: The number of ILB’s the bank can issue changes in time to maturity and first becomes positive for approx. 6.5. However, the equilibrium $N$ will never lead to an expected constant price level (red line is always above the blue line).
Some simulation

Figure: with slightly other parameters the situation changes dramatically. When issuing the equilibrium $N$ ILB’s with time to maturity of about 3 we observe decreasing expected price level.