Exponential Hedging under Variable Horizons

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6th World Congress: Bachelier Finance Society
June 23–26, 2010, Toronto, Canada

This talk is based on joint work with Junfeng Ma (U of A) and Marie-Amelie Morlais (Lemans)
Main points of the talk

The ultimate goal & our approach’s intuition
Market Model, Definitions and Notations
Main Problems: Statement
Horizon-unbiased vs Forward utility
Explicit description of Exponential Forward utilities
Horizon-unbiased exponential hedging
Structures of the horizon-dependent optimal wealth
Optimal Sale problem
Minimal entropy-Hellinger local densities and its varieties
References

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- The mathematical/economical/financial model
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- Answers(partial) to the main problems
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- The mathematical/economical/financial model
- Main problems issued by our approach
- Answers(partial) to the main problems
- Main Mathematical/Statistical Tool: Minimal Hellinger Local Martingale Densities
Main Goal and our approach

Consider an investor endowed with a utility and facing a payoff, $H$, at the maturity time $T$.

$$\max_{\theta \text{ admissible}} EU \left( x + G_T(\theta) - H \right).$$
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Consider an investor endowed with a utility and facing a payoff, $H$, at the maturity time $T$.

$$\max_{\theta \text{ admissible}} \mathbb{E} U (x + G_T(\theta) - H).$$

Let $\tau$ be a nonnegative random variable bounded by $T$. Then, More realistically,

$$\max_{\theta \text{ admissible}} \mathbb{E} U (x + G_\tau(\theta) - H).$$
Generally we can look at

\[
\max_{\theta \text{ admissible}} EU \left( x + G_{\tau}(\theta) - H_{\tau} \right),
\]

where \( H = (H_t) \) is an American-type payoff.
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\max_{\theta \text{ admissible}} \quad EU \left( x + G_{\tau}(\theta) - H_{\tau} \right),
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We want to know how the optimal hedging strategy or the corresponding optimal wealth process is affected by the random horizon (the additional randomness in the horizon, the length of the horizon, etecetera).
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We want to know how the optimal hedging strategy or the corresponding optimal wealth process is affected by the random horizon (the additional randomness in the horizon, the length of the horizon, etcetera).


Our approach: Consider a "nice" random time $\tau$,

\[
EU(x + G(\theta)_{\tau} - H_{\tau}) = E \int_0^{+\infty} U(x + G(\theta)_t - H_t) dF_t \\
= \int_0^{\infty} EU(x + G(\theta)_{\tau_t} - H_{\tau_t}) dt.
\]
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Which $H$ for which the optimal strategy does not depend on the horizon stopping times? and what we can say about this optimal strategy? This is the horizon-unbiased hedging
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How far is problem is close to the self-generating utilities (the forward utilities introduced by Musiela and Zariphoupoulou)...
The market model

$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, P)$ satisfies the usual conditions.
The market model

- \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, P)\) satisfies the usual conditions.
- \(S = (S_t)_{0 \leq t \leq T}\) = assets’ discounted price process
  = a \(d\)-dimensional locally bounded semi-martingale.
The market model

\begin{itemize}
  \item \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, P)\) satisfies the usual conditions.
  \item \(S = (S_t)_{0 \leq t \leq T}\) = assets’ discounted price process
  \begin{align*}
    &= \text{a } d\text{-dimensional locally bounded semi-martingale.}
    \\
    \mathcal{M}_f^e(S) := \{ Q \mid Q \sim P, \quad S \text{ is a } Q\text{ – local martingale,} \\
    \quad H(Q|P) = E \left[ \frac{dQ}{dP} \log \left( \frac{dQ}{dP} \right) \right] < +\infty \} \neq \emptyset
  \end{align*}
\end{itemize}
The market model

$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, P)$ satisfies the usual conditions.

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= a $d$-dimensional locally bounded semi-martingale.

$\mathcal{M}_f^e(S) := \{ Q \mid Q \sim P, \text{ } S \text{ is a } Q - \text{ local martingale, }\$

$\quad H(Q|P) = E \left[ \frac{dQ}{dP} \log \left( \frac{dQ}{dP} \right) \right] < +\infty \} \neq \emptyset$

Consider a semimartingale $B$ such that there exists $p > 1$,

$$\sup_{\tau \in T} E(e^{pB_\tau}) < +\infty. \quad (1)$$
Suppose that the dynamic claim $B$ is bounded from below and satisfies (1), then we denote

$$\Theta(S, B) = \text{the set of admissible strategies} \quad (2)$$

Similar admissibility as in Delbean et al. (2002) or Kabanov and Stricker (2002), but for any stopping time.

See the works of Musiela/Zariphoupoulou since early 2003, the paper of Hobson/Henderson (2007), Choulli/Stricker (2005, 2006), and Choulli/Stricker/Li(2007).
Suppose that the dynamic claim $B$ is bounded from below and satisfies (1), then we denote

$$\Theta(S, B) = \text{the set of admissible strategies}$$ \hspace{1cm} (2)

Similar admissibility as in Delbean et al. (2002) or Kabanov and Stricker (2002), but for any stopping time.

A random field utility $U(t, \omega, x) = U(t, x)$ is a forward utility if there exists "an admissible strategy" $\tilde{\theta}$ such that $U(t, x + (\tilde{\theta} \cdot S)_t)$ is a true martingale, and for any admissible strategy $\theta$, $U(t, x + (\theta \cdot S)_t)$ is a supermartingale.

See the works of Musiela/Zariphopoulou since early 2003, the paper of Hobson/Henderson (2007), Choulli/Stricker (2005, 2006), and Choulli/Stricker/Li(2007).
Main Problems

- **Exponential Forward utilities** Explicit description of semimartingales $B$ such that the following utility
  \[ U(t, \omega, x) = -\exp(-x + B_t(\omega)), \]
  is a forward utility.
Main Problems

- **Exponential Forward utilities** Explicit description of semimartingales $B$ such that the following utility
  \[ U(t, \omega, x) = -\exp(-x + B_t(\omega)), \]
  is a forward utility.

- **Horizon-unbiased exponential hedging** Explicit description of semimartingale $B$ for which we can find
  \( \hat{\theta} \in \Theta(S, B) \) such that for any \( \tau \in T_b \),

  \[
  \max_{\theta \in \Theta(S, B)} E \left\{ -\exp \left( B_\tau - (\theta \cdot S)_\tau \right) \right\} = E \left\{ -\exp \left( B_\tau - (\hat{\theta} \cdot S)_\tau \right) \right\}.
  \]

  (3)

  Explicit description of the strategy $\hat{\theta}$. 

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Exponential Hedging under Variable Horizons
Horizon-dependence structure in optimal strategy/wealth Given bounded semimartingale, \( B \), for which there exists a two-parameters process \( \left( \hat{\theta}(t, s) \right)_{0 \leq s, t \leq T} \), such that for ant \( t \in (0, T] \), \( \theta(t, \cdot) \) is solution

\[
\max_{\theta \in \Theta(S, B)} E - \exp \left[ B_t - G_t(\theta) \right].
\]

Determine the structures of the optimal wealth over horizon, i.e., \( \hat{X}_t := \int_0^t \hat{\theta}(t, u)dS_u, \ 0 \leq t \leq T \).
- Horizon-dependence structure in optimal strategy/wealth: Given bounded semimartingale, $B$, for which there exists a two-parameters process $(\hat{\theta}(t, s))_{0 \leq s, t \leq T}$, such that for any $t \in (0, T]$, $\theta(t, \cdot)$ is solution

$$\max_{\theta \in \Theta(S, B)} E - \exp [B_t - G_t(\theta)].$$

Determine the structures of the optimal wealth over horizon, i.e., $\hat{X}_t := \int_0^t \hat{\theta}(t, u) dS_u$, $0 \leq t \leq T$.

- Optimal Sale Problem: Given $B$ find/describe -as explicit as possible- $\tilde{\theta}$ and $\tau^*$ s.t.

$$\inf_{\tau \in \mathcal{T}} \inf_{\theta \in \Theta(S, B)} E\exp \left[ B_{\tau} - G_{\tau}(\theta) \right] = E\exp \left[ B_{\tau^*} - G_{\tau^*}(\tilde{\theta}) \right].$$
Horizon-unbiased vs Forward utility

**Proposition:** The following are equivalent:

- The following utility \( U(t, \omega, x) = -\exp(-x + B_t(\omega)) \), is a forward utility.
Horizon-unbiased vs Forward utility

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- The following utility $U(t, \omega, x) = -\exp(-x + B_t(\omega))$, is a forward utility.

- The horizon-unbiased problem in (3) admits a solution and the optimal value function, $u(\tau)$, does not depend on the horizon $\tau$. That is $u(\tau) = u(0), \forall \tau \in \mathcal{T}$. 

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Exponential Hedging under Variable Horizons
Explicit description of Exponential Forward utilities
**The predictable case**

**Theorem 1:** Consider a bounded predictable process with finite variation denoted by $B$. Then, the following assertions are equivalent:

1. The random utility
   \[ U(t, \omega, x) = -\exp(-x + B_t(\omega)), \]
   is a forward utility.
The predictable case

**Theorem 1:** Consider a bounded predictable process with finite variation denoted by $B$. Then, the following assertions are equivalent:

- (i) The random utility
  \[ U(t, \omega, x) = -\exp(-x + B_t(\omega)), \]
  is a forward utility.

- (ii) The Minimal entropy-Hellinger martingale measure, $\tilde{Q}$, exists and
  \[ B = B_0 + h^E(\tilde{Q}, P). \]
Through out the remaining part of this section, we suppose given a semimartingale, $B$, that is bounded from below and satisfying (1). Consider the Multiplicative Doob-Meyer decomposition

$$e^B = \mathcal{E}(M^{(B)})e^{B'}.$$

(4)

$M^{(B)}$ is a local martingale and $B'$ is a predictable process with finite variation and $B'_0 = B_0$.

For a numéraire, $N$, given by $N := \mathcal{E}(\pi \cdot S) \ (\pi \in L(S))$, we associate the new discounted asset prices process

$$\bar{S} := S - \frac{\pi}{1 + \pi^* \Delta S} \cdot [S, S].$$

(5)
**Theorem 2:** The following assertions are equivalent:

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Theorem 2: The following assertions are equivalent:

- (i) The random utility \( U(t, \omega, x) := - \exp(-x + B_t(\omega)) \), is a forward utility.
- (ii) The following properties are fulfilled:
  - (a) The MEHM local martingale density with respect to \( \mathcal{E}(M^{(B)}) \) exists, denoted by \( \tilde{Z}^{(B)} \), and satisfies

\[
B - B_0 = \log \left[ \mathcal{E}(M^{(B)}) \right] + h^E \left( \tilde{Z}^{(B)} | \mathcal{E}(M^{(B)}) \right). \quad (6)
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\]

(b) The process \( \tilde{Z}^{(B)} := \mathcal{E}(M^{(B)}) \tilde{Z}^{(B)} \) represents the density of a martingale measure, \( \hat{Q}^{(B)} \), that has finite \( P \)-entropy, i.e., \( H \left( \hat{Q}^{(B)} \big| P \right) := E \left\{ \tilde{Z}^{(B)} \log \left[ \tilde{Z}^{(B)} \right] \right\} < +\infty \).
The problem

Consider a bounded from below semimartingale, $B$ satisfying (1) and a numéraire, $N$, that we suppose bounded away from zero and from above (to simplify). We are interested in addressing our four problems for the utility function:

$$U(t, \omega, x) = -\exp \left[-\frac{x - B_t(\omega)}{N_t(\omega)}\right].$$
Transformation for Horizon-Unbiased Problem

**Theorem 3:** Then following assertions are equivalent:

1. There exists a strategy $\tilde{\theta} \in \Theta(S, B)$ s.t. $\forall \tau \in T_b$,

   $$
   \min_{\theta \in \Theta(S, B)} E^{exp} \left[ \frac{1}{N_{\tau}} \left( B_{\tau} - (\theta \cdot S)_\tau \right) \right] = E^{exp} \left[ \frac{1}{N_{\tau}} \left( B_{\tau} - (\tilde{\theta} \cdot S)_\tau \right) \right].
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1. There exists a strategy \( \tilde{\theta} \in \Theta(S, B) \) s.t. \( \forall \tau \in T_b \),
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   \]
   \( (7) \)

2. There exists a strategy \( \tilde{\varphi} \in \Theta(\overline{S}, \frac{B}{N}) \) s.t. \( \forall \tau \in T_b \),
   \[
   \min_{\varphi \in \Theta(\overline{S}, \frac{B}{N})} E^{\text{exp}} \left[ \frac{B_\tau}{N_\tau} - (\varphi \cdot \overline{S})_\tau \right] = E^{\text{exp}} \left[ \frac{B_\tau}{N_\tau} - (\tilde{\varphi} \cdot \overline{S})_\tau \right].
   \]
   \( (8) \)

Furthermore, \( \tilde{\theta} \) and \( \tilde{\varphi} \) satisfy
\[
\tilde{\varphi} = (\tilde{\theta} \cdot S) - \pi - \tilde{\theta}.
\]
Theorem 3: Then following assertions are equivalent:

(i) There exists a strategy $\tilde{\theta} \in \Theta(S, B)$ s.t. $\forall \tau \in T_b$,

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\min_{\theta \in \Theta(S, B)} E^{exp} \left[ \frac{1}{N_T} \left( B_{\tau} - (\theta \cdot S)_{\tau} \right) \right] = E^{exp} \left[ \frac{1}{N_T} \left( B_{\tau} - (\tilde{\theta} \cdot S)_{\tau} \right) \right].
$$

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(ii) There exists a strategy $\tilde{\varphi} \in \Theta(\overline{S}, \frac{B}{N})$ s.t. $\forall \tau \in T_b$,

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\min_{\varphi \in \Theta(\overline{S}, \frac{B}{N})} E^{exp} \left[ \frac{B_{\tau}}{N_T} - (\varphi \cdot \overline{S})_{\tau} \right] = E^{exp} \left[ \frac{B_{\tau}}{N_T} - (\tilde{\varphi} \cdot \overline{S})_{\tau} \right].
$$

\[\text{(8)}\]

Furthermore, $\tilde{\theta}$ and $\tilde{\varphi}$ satisfy $\tilde{\varphi} = \left( \tilde{\theta} \cdot S \right) \pi - \tilde{\theta}$. 
Theorem 4:
The random utility functional,

\[ U(t, \omega, x) = -\exp \left( -\frac{x - B_t(\omega)}{N_t(\omega)} \right), \]

is a forward utility performance for \( S \) if and only if the random utility

\[ \bar{U}(t, \omega, x) = -\exp \left[ -x + \frac{B_t(\omega)}{N_t(\omega)} \right], \]

is a forward utility performance for \( \bar{S} \) defined in (5).
Horizon-unbiased exponential hedging
Predictable Dynamic Claim

**Theorem 5:**

Let $B$ be a bounded predictable process with finite variation, for which there exists $\hat{\theta} \in \Theta(S, B)$ and for any $\tau \in \mathcal{T}$, (3) holds. That is

$$
\max_{\theta \in \Theta(S, B)} -E \left[ \exp \left( B_\tau - (\theta \cdot S)_\tau \right) \right] = -E \left[ \exp \left( B_\tau - (\hat{\theta} \cdot S)_\tau \right) \right].
$$

(9)

Then the following hold:
Predictable Dynamic Claim

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Then the following hold:

(i) $\hat{\theta}$ coincides with $-\tilde{\theta}^{MEH}$ defined in (12).
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(9)

Then the following hold:

(i) $\hat{\theta}$ coincides with $-\tilde{\theta}_{MEH}^\theta$ defined in (12).

(ii) The processes $B$ and $h^E(\tilde{Z}, P)$ coincide on

$$\{(t, \omega) \mid \text{there exists } \theta \in \Theta(S, B) \text{ such that } (\theta \cdot S)_{t-}(\omega) \neq 0\}.$$
**Theorem 6:** Consider a bounded from below semimartingale, $B$, satisfying (1), and consider the processes $M^{(B)}$ and $B'$ given in (4). Suppose that there exists $\hat{\theta} \in \Theta(S, B)$ such that for any $\tau \in T_b,$

$$\max_{\theta \in \Theta(S, B)} -E \left[ \exp \left( B_{\tau} - (\theta \cdot S)_{\tau} \right) \right] = -E \left[ \exp \left( B_{\tau} - (\hat{\theta} \cdot S)_{\tau} \right) \right].$$

(10)

Then, the following assertions are fulfilled:

- (a) The process $\hat{\theta}$ coincides with $-\tilde{\theta}^{MEH}(B)$ given by the decomposition of the MEHM local martingale density, $\tilde{Z}(B)$, with respect to $\mathcal{E}(M^{(B)})$ via the equation (12).
Theorem 6: Consider a bounded from below semimartingale, $B$, satisfying (1), and consider the processes $M(B)$ and $B'$ given in (4). Suppose that there exists $\hat{\theta} \in \Theta(S, B)$ such that for any $\tau \in T_b$,

$$\max_{\theta \in \Theta(S, B)} -E \left[ \exp \left( B_\tau - (\theta \cdot S)_\tau \right) \right] = -E \left[ \exp \left( B_\tau - (\hat{\theta} \cdot S)_\tau \right) \right].$$

(10)

Then, the following assertions are fulfilled:

- (a) The process $\hat{\theta}$ coincides with $-\tilde{\theta}^{MEH}(B)$ given by the decomposition of the MEHM local martingale density, $\tilde{Z}(B)$, with respect to $\mathcal{E}(M(B))$ via the equation (12).

- (b) The two processes $B'$ and $h^E(\tilde{Z}(B)|M(B))$ coincide on

$$\{(t, \omega) \mid \text{there exists } \theta \in \Theta(S, B) \text{ such that } (\theta \cdot S)_{t-}(\omega) \neq 0\}.$$
**Theorem 7:** For any $t \in (0, T]$, $\hat{X}_t := \int_0^t \hat{\theta}(t, u) dS_u$ is the optimal wealth in exponentially hedging the claim $B_t$. Then the following assertions hold:

1. (i) The process $\hat{X}$ has a càdlàg modification that is a semimartingale.

Choulli and Schweizer (2009).
**Theorem 7:** For any $t \in (0, T]$, $\hat{X}_t := \int_0^t \hat{\theta}(t, u) dS_u$ is the optimal wealth in exponentially hedging the claim $B_t$. Then the following assertions hold:

- (i) The process $\hat{X}$ has a càdlàg modification that is a semimartingale.
- (ii) The two-parameters process $\left(\hat{\theta}(t, s)\right)_{0 \leq s, t \leq T}$ has version that has finite variation in the first parameter $t$, and its diagonal $\hat{\theta}_t := \hat{\theta}(t, t)$ is $S$-integrable.

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(iii) The process $\hat{X} - (\hat{\theta} \cdot S)$ is predictable with finite variation. Choulli and Schweizer (2009).
Optimal Sale problem
Main Results

- The process

\[ V(t) := \text{ess sup}_{\tau \geq t, \theta \in \Theta} E \left( -\exp \left[ -\int_t^\tau \theta_u dS_u + B_\tau \right] | F_t \right). \]

admits a càdlàg modification and satisfies

\[ V(t) = \max \left[ -e^{B_t}; \text{ess sup}_{\tau > t, \theta \in \Theta} E \left( V_\tau e^{-\int_t^\tau \theta_u dS_u} | F_t \right) \right]. \]
Main Results

- The process

\[ V(t) := \text{ess sup}_{\tau \geq t, \theta \in \Theta} E \left( -\exp \left[ -\int_t^\tau \theta_u dS_u + B_\tau \right] | \mathcal{F}_t \right). \]

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\[ V(t) = \max \left[ -e^{B_t}; \text{ess sup}_{\tau > t, \theta \in \Theta} E \left( V_\tau e^{-\int_t^\tau \theta_u dS_u} | \mathcal{F}_t \right) \right]. \]

- \( V \) is a càdlàg negative semimartingale s.t.

\[ V(t) = V(0)\mathcal{E}(M^V)e^{A^V}, \quad M^V = \beta \cdot S^c + (f-1) \star (\mu - \nu) + g \star \mu + M'. \]

Here \( A^V \) is a predictable process with finite variation, \( M^V \) is a local martingale, and \((\beta, f, g, M')\) are the Jacod’s components for \( M^V \).

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Exponential Hedging under Variable Horizons
The optimal investment time is given by

$$\tau^* = \tau_0 := \inf \{0 \leq t \leq T \mid V(t) = -e^{B_t} \text{ or } V(t-) = -e^{B_{t-}} \} \land T.$$ 

That is $$V(0) = \sup_{\theta \in \Theta} E \left[ -e^{-\langle \theta, S \rangle_{\tau^*} + B_{\tau^*}} \right],$$ or more generally

$$V(t) = \text{ess sup}_{\theta \in \Theta} E \left( -\exp \left[ -\int_{t}^{T} \theta_u dS_u + B_{\tau_t} \right] | \mathcal{F}_t \right),$$

$$\tau_t := \inf \{u \in [t, T] \mid V(u) = -e^{B_u} \text{ or } V(t-) = -e^{B_{t-}} \} \land T.$$
If the optimal strategy $\theta^*$ exists, then on $[0, \tau^*]$ it is the pointwise root of

$$0 = b + c(\theta - \beta) + \int \left[1 - f(x)e^{-\theta^T x}\right] xF(dx).$$
Proposition 1: Consider a semimartingale, $B$ satisfying assumptions of Theorem 6. Then the following assertions hold:

(i) If $h E(\tilde{Z}(B), E(M(B))) \preceq B'$, then the optimal investment time $\tau^*\approx_0$. That is the best option to invest is to liquidate the real asset immediately.

(ii) If $B' \preceq h E(\tilde{Z}(B), E(M(B)))$, then the optimal investment time $\tau^*$ coincides with the horizon $T$. That is the best option to invest is to wait to the maturity to liquidate the real asset.
Proposition 1: Consider a semimartingale, $B$ satisfying assumptions of Theorem 6. Then the following assertions hold:

(i) If $h^E \left(\tilde{Z}(B), \mathcal{E}(M(B))\right) \preceq B'$, then the optimal investment time $\tau^*$ coincides with the initial time 0. That is the best option to invest is to liquidate the real asset immediately.
Particular Cases

**Proposition 1:** Consider a semimartingale, $B$ satisfying assumptions of Theorem 6. Then the following assertions hold:

(i) If $h^E \left( \tilde{Z}(B), \mathcal{E}(M^{(B)}) \right) \preceq B'$, then the optimal investment time $\tau^*$ coincides with the initial time 0. That is the best option to invest is to liquidate the real asset immediately.

(ii) If $B' \preceq h^E \left( \tilde{Z}(B), \mathcal{E}(M^{(B)}) \right)$, then the optimal investment time $\tau^*$ coincides with the horizon $T$. That is the best option to invest is to wait to the maturity to liquidate the real asset.
Proposition 2: Suppose that \( B' - h^E \left( \tilde{Z}^{(B)}, \mathcal{E}(M^{(B)}) \right) \) increases on \((\tau_{2i}, \tau_{2i+1})\) and decreases on \((\tau_{2i+1}, \tau_{2i+2})\), \(i = 0, ..., N\), then the quantity \(\inf_{\tau \in T} \inf_{\theta \in \Theta(S,B)} E e^{B_\tau - G_\tau(\theta)}\) coincides with

\[
\min\left\{ \inf_{\theta \in \Theta(S,B)} E \exp \left[ B_{T_{2i+1}} - G_{T_{2i+1}}(\theta) \right], \ i = 0, 1, ..., N \right\}. 
\]
Minimal Hellinger martingale densities
Let \( N \in \mathcal{M}_0, \ loc(P) \) such that \( 1 + \Delta N \geq 0 \). If

\[
V_t^{(E)}(N) := \frac{1}{2} \langle N^c \rangle_t + \sum_{0 < s \leq t} [(1 + \Delta N_s) \log(1 + \Delta N_s) - \Delta N_s]
\]

is locally integrable, then its compensator (with respect to the probability \( P \)) is called the entropy- Hellinger process of \( N \), and is denoted by \( H^{(E)}(N, P) \).

Let \( Q \in \mathcal{P}_a \) with density \( Z = \mathcal{E}(N) \). If \( V_t^{(E)}(N) \) is locally integrable, then its compensator is called the entropy-Hellinger process of \( Q \) with respect to \( P \), and we denoted by

\[
h_t^{(E)}(Q, P) := h_t^{(E)}(Z, P) := H_t^{(E)}(N, P), \quad 0 \leq t \leq T.
\]
We call minimal entropy-Hellinger martingale measure, \( \tilde{Q} \), is the martingale measure -when it exists- solution to

\[
\min_{Q \in \mathcal{M}_f^e(S)} h^E(Q, P),
\]

with respect to the following order: \( X \preceq Y \) if \( Y - X \) is nondecreasing.

When minimizing the densities over the set of local martingale densities \( Z \) that are \( Z \log(Z) \) locally integrable, we obtain the minimal entropy-Hellinger local martingale density, that we denote \( \tilde{Z} \). It is explicitly calculated and is given by

\[
\log \left[ \tilde{Z} \right] = \tilde{\theta}^{MEH} \cdot S + h^E(\tilde{Z}, P). \tag{12}
\]

The predictable processes \( \tilde{\theta}^{MEH} \) and \( h^E(\tilde{Z}, P) \) are explicitly...
Let $Q$ be a probability measure and $Y$ is a $Q$-local martingale such that $1 + \Delta Y \geq 0$. Then if the càdlàg nondecreasing process

$$V^E(Y) = \frac{1}{2} \langle Y^c \rangle + \sum [(1 + \Delta Y) \log(1 + \Delta Y) - \Delta Y], \quad (13)$$

is $Q$-locally integrable (i.e. $V^E(Y) \in A_{loc}^+(Q)$), then its $Q$-compensator is called the entropy-Hellinger process of $Y$ (or equivalently of $\mathcal{E}(Y)$) with respect to $Q$, and is denoted by $h^E(Y, Q)$ (respectively $h^E(\mathcal{E}(Y), Q)$).
Let \( N, \overline{N} \in \mathcal{M}_0, \text{loc}(P) \) such that \( 1 + \Delta N \geq 0 \) and \( 1 + \Delta \overline{N} \geq 0 \). If the càdlàg nondecreasing process

\[
\frac{1}{2} \langle N^c - \overline{N}^c \rangle + \sum \left\{ (1 + \Delta N) \log \left[ \frac{1 + \Delta N}{1 + \Delta \overline{N}} \right] - \Delta N + \Delta \overline{N} \right\},
\]

is locally integrable, then its compensator is called the entropy-Hellinger process of \( \mathcal{E}(N) \) with respect to \( \mathcal{E}(\overline{N}) \), and is denoted by \( h^E (\mathcal{E}(N) | \mathcal{E}(\overline{N})) \).

If \( Q_n := \mathcal{E}_{T_n}(\overline{N}) \cdot P \), then

\[
h^E_{t \wedge T_n} (\mathcal{E}(N) | \mathcal{E}(\overline{N})) = h^E_{t \wedge T_n} (\mathcal{E}(N), Q_n).
\]
Principal Works

- Choulli, T., Stricker, Ch., and Li, Jia (2007): ”Minimal Hellinger martingale measure of order $q$”, To appear in *Finance and Stochastics*. 


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• **Kabanov, and Stricker (2002):** ”On the optimal portfolio for the exponential utility maximization: Remarks to the six-author paper”, Mathematical Finance.

Musiela, M. and Zariphopoulou, T.: See the numerous works of these authors including ”Investments and Forward utilities” and ”Investment and Valuation under backward and Forward dynamic utilities in a stochastic factor model”.

Thank you for your attention