Mean Variance Optimization with State Dependent Risk Aversion

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The Classical Mean Variance Problem

- $X_t$ - wealth process
- $u_t$ - amount invested in the risky asset

$$\max E_t, x [X_T] - \frac{\gamma}{2} \text{Var}_t, x [X_T]$$

- time inconsistency

$$\max E_t, x [F(X_T)] + G (E_t, x [X_T])$$

- the control does not depend on the current wealth

$$u(t, x) = h(t)$$
Why State Dependent Risk Aversion

- $u(t, x) = h(t)$ since the risk aversion parameter $\gamma$ is constant.
- It does not matter if the wealth is 100 USD, or 100 000 000 USD, we invest the same amount of dollars in the risky asset.
- Conceptual difference between the 1-period model and the multi-period model.
- Make $\gamma$ explicitly dependent on $X_t$

$$\max E_{t,x}[X_T] - \frac{\gamma(x)}{2} \text{Var}_{t,x}[X_T]$$

Where $X_t = x$
riskless asset

\[ dB_t = rB_t \, dt \]

risky asset

\[ dS_t = \alpha S_t \, dt + \sigma S_t \, dW_t \]

wealth portfolio

\[ dX_t = [rX_t + (\alpha - r)u_t] \, dt + \sigma u_t \, dW_t \]

\( u_t \) - amount of money invested in the stock

\[ J(t, x, u) = E_{t,x}[X_T] - \frac{\gamma(x)}{2} \, Var_{t,x}[X_T] \]
Time Inconsistency

\[ E_{t,x}[X_T] - \frac{\gamma(x)}{2} \text{Var}_{t,x}[X_T] = E_{t,x}[F(x, X_T)] + G(x, E_{t,x}[X_T]) \]

- standard dynamic programming problem \( \max E_{t,x}[F(X_T)] \)
- here \( \max E_{t,x}[F(x, X_T)] + G(x, E_{t,x}[X_T]) \)
- conceptual problem - what is optimal?
- computational problem - how we compute it?
Possible ways out

- **Pre-commitment:** Solve (somehow) the problem at 0, $x_0$ and ignore the fact that later on, your “optimal” control will no longer be viewed as optimal.

- **Game theory:** Take the time inconsistency seriously. View the problems as a game and look for a Nash equilibrium point.

  - Ekeland & Lazrak (2006); Ekeland & Pirvu (2007)
  - Basak & Chabakauri (2008)
  - Björk & Murgoci (2009)
The Game Theoretic Approach

- We view this as a game where there is one player for each $t$.
- Player No $t$ chooses the control function $u(t, \cdot)$ at time $t$, and applies the control $u(t, X_t)$.
- The value, to player No $t$, if all players use the control law $u$ is

$$J(t, x; u) = E_{t,x} [x, F(X^u_T)] + G(x, E_{t,x}[X^u_T])$$
Subperfect Nash Equilibrium

**Definition**

The strategy $\hat{u}$ is a **Nash subgame perfect equilibrium** if the following holds for all $t$:

- Assume that all players No $s$ with $s > t$ use the control $\hat{u}(s, X_s)$.
- Then it is optimal for player No $t$ also to use $\hat{u}(t, X_t)$.

**Note!**

- this leads to an extension of the HJB equation as a PDE system with an embedded fixed point problem.
Motivation  Our Problem  Time Inconsistency  Numerical Results  Conclusions

Notation

\[ V(T, x) = F(x, x) + G(x, x) \]
\[ F(x, y) = y - \frac{\gamma(x)}{2} y^2, \]
\[ G(x, y) = \frac{\gamma(x)}{2} y^2. \]

- Probabilistic interpretation

\[ f(t, x, y) = E_{t,x} \left[ F(y, X_T^u) \right] \]
\[ g(t, x) = E_{t,x} \left[ X_T^u \right] \]

Note! \[ V(t, x) = f(t, x, x) + \frac{\gamma(x)}{2} g^2(t, x) \]
Fixed Point PDE System

\[
\sup_{u \in \mathcal{U}} \left\{ (\mathcal{A}^u V)(t, x) - (\mathcal{A}^u f)(t, x, x) + (\mathcal{A}^u f^x)(t, x) \right. \\
\left. - \mathcal{A}^u G(x, g(t, x)) + G_y(x, g(t, x)) \cdot \mathcal{A}^u g(t, x) \right\} = 0,
\]

\[
\mathcal{A}^\hat{u} f^y(t, x) = 0,
\]

\[
\mathcal{A}^\hat{u} g(t, x) = 0,
\]

\[
V(T, x) = F(x, x) + G(x, x),
\]

\[
f(T, x, y) = F(y, x),
\]

\[
g(T, x) = x.
\]

Remember! \( V(t, x) = f(t, x, x) + \frac{\gamma(x)}{2} g^2(t, x) \)
Solving the PDE system

Optimal control

\[
\hat{u}(t, x) = -\alpha - r \frac{f_x(t, x, x) + \gamma(x)g(t, x)g_x(t, x)}{\sigma^2 f_{xx}(t, x, x) + \gamma(x)g(t, x)g_{xx}(t, x)}
\]

New PDE system

\[
f_t + [rx + (\alpha - r)\hat{u}]f_x + \frac{1}{2}\sigma^2 f_{xx} = 0
\]
\[
g_t + [rx + (\alpha - r)\hat{u}]g_x + \frac{1}{2}\sigma^2 g_{xx} = 0
\]

with \(f\) and \(g\) evaluated at \((t, x, y)\) and

\[
f(T, x, y) = x - \frac{\gamma(y)}{2}x^2
\]
\[
g(T, x, y) = x
\]
One Possible Solution

for

\[
\gamma(x) = \frac{\gamma}{x}
\]

we show that

- \( \hat{u}(t, x) = c(t)x \) is a solution to the PDE system
- \( c(t) = \frac{\beta}{\gamma \sigma^2} \left[ \frac{a(t)}{b(t)} + \gamma \left( \frac{a^2(t)}{b(t)} - 1 \right) \right] \) where

\[
\beta = \alpha - r
\]

\[
a(t) = e^{\int_t^T [r + \beta c(s)] ds}
\]

\[
b(t) = e^{2 \int_t^T [r + \beta c(s) + \frac{1}{2} \sigma^2 c^2(s)] ds}
\]

- \( V(t, x) = \{ a(t) + \frac{\gamma}{2} [a^2(t) - b(t)] \} x \)
Existence for $c(t)$

$$
c_0(t) = 1
$$

$$
c_{n+1}(t) = \frac{\beta}{\gamma \sigma^2} \left[ e^{-\int_t^T [r + \beta c_n(s) + \sigma^2 c_n^2(s)] ds} + \gamma e^{-\int_t^T \sigma^2 c_n^2(s) ds} - \gamma \right], \quad n = 0, 1, 2...
$$

- **Step 1.** $\{c_n(\cdot)\}$ uniformly bounded in $C([0, T])$
- **Step 2.** $\{\dot{c}_n(\cdot)\}$ uniformly bounded in $C([0, T])$.
- **Step 3.** For any $t_1, t_2 \in [0, T]$, we have

$$
|c_n(t_2) - c_n(t_1)| = \int_0^1 \dot{c}_n(t_1 + \theta(t_2 - t_1)) d\theta(t_2 - t_1) \leq k|t_2 - t_1| \quad \forall n
$$

where $k$ is a constant independent of $n$. 
Existence for $c(t)$

\[
c_0(t) = 1
\]
\[
c_{n+1}(t) = \frac{\beta}{\gamma \sigma^2} \left[ e^{-\int_t^T [r + \beta c_n(s) + \sigma^2 c_n^2(s)] ds} + \gamma e^{-\int_t^T \sigma^2 c_n^2(s) ds} - \gamma \right],
\]
\[
n = 0, 1, 2, ...
\]

Step 1+2+3 $\Rightarrow$ there is a $c(\cdot) \in C([0, T])$ such that
\[
c_{n_i}(\cdot) \xrightarrow{n \to \infty} c(\cdot) \in C([0, T])
\]

Uniqueness can be proved easily.
Proportion of Money invested in the Stock for Various $\gamma$
Proportion of Money invested in the Stock for Various Time Horizons
THANK YOU!