On the dual problem associated to the robust utility maximization in a market model driven by a Lévy Process

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Starting from an expected utility problem of the kind

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It is common use to denote (1) as the Merton-problem

Pliska provided the martingale and duality approach


The papers


The primal problem

\[
 u_Q(x) := \sup_{X \in \mathcal{X}(x)} \left\{ \mathbb{E}_Q[U(X_T)] \right\}. \tag{2}
\]

over a set of admissible wealth processes \( \mathcal{X}(x) \), lead to the dual value function

\[
 v_Q(y) := \inf_{Y \in \mathcal{Y}_Q(y)} \left\{ \mathbb{E}_Q[V(Y_T)] \right\}. \tag{3}
\]
• Gilboa, I. & Schmeidler, D. 1989 “Maxmin expected utility with a non-unique prior”, Journal of Mathematical Economics, pp. 141-153. Introduced the "certainty-independence" axiom what lead to robust utility functionals

\[ X \rightarrow \inf_{Q \in Q} \{ \mathbb{E}_Q [U(X)] \} , \]  

(4)

where the set of “prior” models \( Q \) is assumed to be a convex set of probability contents on the measurable space \((\Omega, \mathcal{F})\). The corresponding robust utility maximization problem

\[ \inf_{Q \in Q} \{ \mathbb{E}_Q [U(X)] \} \rightarrow \max, \]  

(5)

had being considered by several authors:

The former worst case approach do not discriminate among all the possible models in $Q$, what again is reflected in inconsistencies in the axiom system proposed.


introduced a relaxed axiom system which leads to utility functionals

$$X \rightarrow \inf_{Q \in Q} \{ \mathbb{E}_{Q}[U(X)] + \vartheta(Q) \}, \quad (6)$$

where the penalty function $\vartheta$ assigns a weight $\vartheta(Q)$ to each model $Q \in Q$. 

The corresponding dual theory for utility functions defined in the positive halfline

\[ u(x) := \sup_{X \in \mathcal{X}(x)} \inf_{Q \in \mathcal{Q}} \{ \mathbb{E}_Q[U(X_T)] + \vartheta(Q) \} . \quad (7) \]

was developed in


introducing the robust dual value function

\[ v(y) = \inf_{Q \ll \mathcal{Q}} \{ v_Q(y) + \vartheta(Q) \} \]

\[ = \inf_{Q \ll \mathcal{Q}} \inf_{Y \in \mathcal{Y}_Q(y)} \{ \mathbb{E}_Q[V(Y_T)] \} + \vartheta(Q) \}. \quad (8) \]
The Probability Space

- \( \{L_t\}_{t \in \mathbb{R}_+} \) be a Lévy process (i.e. a cádlág process with independent stationary increments starting at zero).
- A filtered probability space \((\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F})\) with \(\mathcal{F} := \{\mathcal{F}_t^\mathbb{P}(L)\}_{t \in \mathbb{R}_+}\) the completion of its natural filtration, i.e.

\[
\mathcal{F}_t^\mathbb{P}(L) := \sigma \{L_s : s \leq t\} \vee \mathcal{N}
\]

where \(\mathcal{N}\) is the \(\sigma\)-algebra generated by all \(\mathbb{P}\)-null sets.
- Further we denote the jump measure of \(L\) by \(\mu : \Omega \times (\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R}_0)) \rightarrow \mathbb{N}\) where \(\mathbb{R}_0 := \mathbb{R} \setminus \{0\}\)
- Recall that its dual predictable projection, also known at its Lévy system, fulfills

\[
\mu^p (dt, dx) = dt \otimes \nu (dx)
\]

where \(\nu(\cdot) := \mathbb{E} [\mu ([0, 1] \times \cdot)].\)
Denote the class of predictable processes $\theta \in \mathcal{P}$ integrable with respect to $U^c$ in the sense of local martingale

$$
\mathcal{L} (U^c) := \left\{ \theta \in \mathcal{P} : \exists \{\tau_n\}_{n \in \mathbb{N}} \text{ sequence of stopping times}
\right.
\left. \text{with } \tau_n \uparrow \infty \text{ and } \mathbb{E} \left[ \int_0^{\tau_n} \theta^2 d [U^c] \right] < \infty \quad \forall n \in \mathbb{N} \right\}
$$

$$
\Lambda (U^c) := \left\{ \int \theta_0 dU^c : \theta_0 \in \mathcal{L} (U^c) \right\}
$$
the linear space of processes which admits a representation as the stochastic integral w.r.t. $U^c$.

We denote by $\mathcal{P} \subset \mathcal{B} (\mathbb{R}_+) \otimes \mathcal{F}$ the predictable $\sigma$-algebra and by

$$
\tilde{\mathcal{P}} := \mathcal{P} \otimes \mathcal{B} (\mathbb{R}_0).
$$

The integral $\int_{\mathbb{R}_0} \theta_1 d (\mu - \mu^P)$ is defined for processes $\theta_1 : \Omega \times \mathbb{R}_+ \times \mathbb{R}_0 \to \mathbb{R}$ of the class

$$
\mathcal{G} (\mu) \equiv \left\{ \theta_1 \in \tilde{\mathcal{P}} : \left\{ \int_{[0,t] \times \mathbb{R}_0} \left\{ \theta_1 (s, x) \right\}^2 \mu (ds, dx) \right\}_{t \in \mathbb{R}_+} \right\}
$$

is adapted increasing loc. integ.
Lemma

For any absolute continuous probability measure $Q \ll P$ there are coefficients $\theta_0 \in \mathcal{L}(W)$ and $\theta_1 \in \mathcal{G}(\mu)$ such that $\frac{dQ_t}{dP_t} = \mathcal{E}(Z^\theta)(t)$ for

$$Z_t^\theta := \int_{0}^{t} \theta_0 \, dW + \int_{0}^{t} \theta_1(s, x) (\mu(ds, dx) - ds \nu(dx)).$$

The coefficients $\theta_0$ and $\theta_1$ are $P$-a.s and $\mu_P(ds, dx)$-a.s. unique respectively.

Notation. We denote the class of absolute continuous probability measure w.r.t. $P$ with

$Q \ll (P)$

and the subclass of equivalent probability measure with

$Q \approx (P)$.

The corresponding classes of density processes for $Q \ll (P)$ and $Q \approx (P)$ is denoted by $D \ll (P)$ and $D \approx (P)$ respectively.
The Market Model

• Let us consider an exogenous factor with a dynamic given by

\[ Y_t := \int_{0,t} \alpha_s ds + \int_{0,t} \beta_s dW_s + \int_{0,t \times \mathbb{R}_0} \gamma(s, x) \left( \mu(ds, dx) - \nu(dx) ds \right), \]

where the processes \( \alpha, \beta, \gamma \) with \( \beta \in \mathcal{L}(W) \) and \( \gamma \in \mathcal{G}(\mu) \) fulfill also the conditions:

(i) \( \int_{0,t} (\alpha_s)^2 ds < \infty \) \( \forall t. \)

(ii) \( \gamma \geq -1 \) \( \mathbb{P} - a.s. \) \( \forall (t, x) \in \mathbb{R}_+ \times \mathbb{R}_0 \)

(iii) \( \gamma \) is a locally bounded process

• The process \( Y \) specifies the discounted price process as its Doleans-Dade exponential

\[ S_t = S_0 \mathcal{E}(Y_t) = S(0) + \int_0^t S_u dY_u, \]
Further let the predictable cadlag process \( \{ \pi_t \}_{t \in \mathbb{R}_+} \) with 
\[
\int_0^t (\pi_s)^2 \, ds < \infty \ \text{P-a.s.} \ \forall t \in \mathbb{R}_+
\]
denotes the proportion of the wealth at time \( t \) invested in the risky asset \( S \) at this time. For an initial capital \( x \) the discounted wealth \( X_t^{x,\pi} \) associated to a self-financing admissible investment strategy \( \pi \) fulfills the equation

\[
X_t^{x,\pi} = x + \int_0^t \frac{X_u^{x,\pi} \pi_u}{S_u} \, dS_u.
\]

An strategy \( \{ \pi_t \}_{t \in \mathbb{R}_+} \) with initial capital \( x \) is called admissible when the wealth process \( X_t^{x,\pi} \geq 0 \ \forall t \) and the class of such wealth processes is denoted by \( \mathcal{X}(x) \).
Our next result characterizes the class of equivalent local martingale measures

\[ Q_{elm} (P) := \{ Q \in \mathcal{Q}_\approx (P) : \mathcal{X} (1) \subset \mathcal{M}_{loc} (Q) \}. \]

**Theorem**

Given \( Q \in \mathcal{Q}_\approx (P) \) let \( \theta_0 \in \mathcal{L} (\mathcal{W}) \), \( \theta_1 \in \mathcal{G} (\mu) \) be the corresponding processes obtained in Lemma 1. Then the following equivalence holds:

\[ Q \in Q_{elm} (P) \iff \alpha_t + \beta_t \theta_0 (t) + \int_{\mathbb{R}_0} \gamma (t, x) \theta_1 (t, x) \nu (dx) = 0 \ \forall \ t \geq 0 \]
Convex measures of risk and the minimal penalty function

- Denote by $Q_{\text{cont}}(\Omega, \mathcal{F})$ the set of **probability contents** on the measurable space $(\Omega, \mathcal{F})$ (i.e. finite additive set functions $Q : \mathcal{F} \to [0, 1]$ with $Q(\Omega) = 1$)
- Let $Q(\Omega, \mathcal{F}) \subset Q_{\text{cont}}(\Omega, \mathcal{F})$ be the family of probability measures.
- From the general theory of convex risk measures, we know that any functional
  \[ \psi : Q_{\text{cont}}(\Omega, \mathcal{F}) \to \mathbb{R} \cup \{+\infty\} \]
  with
  \[ \inf_{Q \in Q_{\text{cont}}} \psi(Q) > -\infty \]
  induce a convex measure of risk as an application
  \[ \rho : \mathcal{M}_b(\Omega, \mathcal{F}) \to \mathbb{R} \]
  given by
  \[ \rho(X) := \sup_{Q \in Q_{\text{cont}}} \{ \mathbb{E}_Q[-X] - \psi(Q) \} . \quad (10) \]
Let now $h_0$ and $h_1$ be $\mathbb{R}_+\text{-valued}$ convex, lower semicontinuous functions with $h_0(0) = 0 = h_1(0)$ which satisfy the conditions

\[
h_0(x) \geq \kappa_1 x^2 - \kappa_2, \quad h_1(x) \geq 2\kappa_1 x \ln(1 + x) \vee |x| \vee |(1 + x) \ln(1 + x)|,
\]

for some constants $\kappa_1, \kappa_2 > 0$. Further define the penalty function

\[
\vartheta(Q) = \mathbb{E}_Q \left[ \int_0^T h_0(\theta_0(t)) \, dt + \int_{[0, T] \times \mathbb{R}_0} h_1(\theta_1(t, x)) \mu^P(dt, dx) \right] 1_{Q_\ll} + \infty \times 1_{Q_{\text{cont}} \setminus Q_\ll}(Q),
\]

where $\theta_0, \theta_1$ are the processes associated to $Q$ from Lemma 1, and the convex measure of risk

\[
\rho(X) := \sup_{Q \in Q_\ll(P)} \left\{ \mathbb{E}_Q[-X] - \vartheta(Q) \right\}.
\]
• Any convex measure of risk \( \rho \) on the space of bounded measurable functions \( \mathcal{M}_b(\Omega, \mathcal{F}) \) is of the form

\[
\rho(X) := \sup_{Q \in Q_{\text{cont}}} \left\{ \mathbb{E}_Q[-X] - \psi^*_\rho(Q) \right\},
\]

where

\[
\psi^*_\rho(Q) = \sup_{X \in A_\rho} \mathbb{E}_Q[-X]
\]

and \( A_\rho := \{ X \in \mathcal{M}_b : \rho(X) \leq 0 \} \) is the acceptance set of \( \rho \). \( \psi^*_\rho(Q) \) is called the **minimal penalty function** associated to \( \rho \) and fulfills the biduality relation

\[
\psi^*_\rho(Q) = \sup_{X \in \mathcal{M}_b(\Omega, \mathcal{F})} \left\{ \mathbb{E}_Q[-X] - \rho(X) \right\} \quad \forall Q \in Q_{\text{cont}}. \tag{13}
\]
Theorem

Let $\psi : \mathcal{Q} \rightarrow \mathbb{R} \cup \{ +\infty \}$ be a function with $\inf_{\mathcal{Q} \in \mathcal{Q}_{\text{cont}}} \psi(Q) > -\infty$ and $\rho(X) := \sup_{\mathcal{Q} \in \mathcal{Q} \langle \mathcal{P} \rangle} \{ \mathbb{E}_Q [-X] - \psi(Q) \}$ the associated convex measure of risk. The penalty $\psi$ is the minimal penalty function associated to $\rho$ i.e. $\psi = \psi_\rho^*$ if $\psi$ is a proper convex function and lower semicontinuous w.r.t. the weak topology $\sigma (L^1, L^\infty)$.

Theorem

The penalty function $\vartheta$ as defined in (11) is the minimal penalty function of the convex risk measure $\rho$ given by (12).
Robust Utility Maximization

- $U : (0, \infty) \rightarrow \mathbb{R}$ is strictly increasing, strictly concave, continuous differentiable, which satisfies the Inada conditions (i.e. $U'(0+) = +\infty$ and $U'(\infty-) = 0$) with asymptotic elasticity strictly less than one.

- Let us now introduce the class

$$
\mathcal{C} := \left\{ \mathcal{E} \left( Z^\xi \right) : \xi := \left( \xi^{(0)}, \xi^{(1)} \right), \xi^{(0)} \in \mathcal{L}(\mathcal{W}), \xi^{(1)} \in \mathcal{G}(\mu), \text{ with } \right. \\
\left. \alpha_t + \beta_t \xi^{(0)}_t + \int_{\mathbb{R}_0} \gamma(t, x) \xi^{(1)}(t, x) \nu(dx) = 0, \ \forall t \right\}
$$

with $Z^\xi$ defined as in (9), and observe that

$$
\mathcal{D}_{elmm}(\mathbb{P}) \subset \mathcal{C} \subset \mathcal{Y}_{\mathbb{P}}(1),
$$

where

$$
\mathcal{Y}_Q(y) := \{ Y \geq 0 : Q\text{-supermartingale}, Y_0 = y, YX \text{ Q}\text{-supermartingale} \}.
$$
If
\[ \nu_Q(y) < \infty \quad \forall Q \in Q^\theta \quad \forall y > 0. \] (14)
we have from Theorem 2 in [Krk&Scha 2003] that
\[ u_Q(x) < \infty \quad \forall Q \in Q^\theta \quad \forall x > 0 \] (15)

**Theorem**

For an utility function \( U \), which fulfills the condition (14), we have that the dual value function turn into

\[ \nu(y) = \inf_{Q \in Q_\leq} \left\{ \inf_{\xi \in \mathcal{C}} \left\{ \mathbb{E}_Q \left[ \nu \left( y \frac{\mathcal{E}(Z^\xi)}{D^Q_T} \right) \right] \right\} + \vartheta(Q) \right\} \] (16)

**Lemma**

For \( U(x) = \log(x) \) we have (14).