Behavioural Portfolio Selection with Loss Control

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Problem setting

- Financial market: complete market with time horizon $T < \infty$
  - Pricing density $\rho$: price of a contingent claim $\xi$ is $E[\rho \xi]$
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  - $S$-shaped utility $u(x) = u_+(x^+) - u_-(x^-)$
  - $u_\pm(\cdot)$ are concave, ↑
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  - S-shaped utility $u(x) = u_+(x^+) - u_-(x^-)$
    - $u_\pm(\cdot)$ are concave, $\uparrow$
  - Probability distortions $T_\pm(\cdot) : [0, 1] \mapsto [0, 1]$
    - $T_\pm \uparrow$, $T_\pm(0) = 0$, $T_\pm(1) = 1$
    - $T_\pm(p) > p$ for small $p$
Problem setting

- Behavioral criterion: for a r.v. $Y$,

$$V(Y) = \int_0^{+\infty} u(y) d[-T_+(P(Y \geq y))] + \int_{-\infty}^0 u(y) d[T_-(P(Y \leq y))]$$
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$$= \int_{0}^{+\infty} T_+(P(u_+(Y^+) \geq y))dy - \int_{0}^{+\infty} T_-(P(u_-(Y^-) \geq y))dy$$
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- Investor’s problem

Maximize $V(X - B)$

s.t.

$$X \in A$$

$$E[X \rho] = x_0$$

where $A$ is the set of admissible terminal wealths.
What is done

• Without probability distortions, the problem was widely studied, like Berkelaar, Kouwenberg and Post (2004)
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  - Jin and Zhou (2008) solved the problem with
    \[ A = \{ X : X \text{ is lower bounded} \} \]
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  ◦ Optimal investment in Jin and Zhou has a deterministic loss in a bad market situation
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- With probability distortion, the problem is much more difficult.
  - Jin and Zhou (2008) solved the problem with:
    \[ A = \{ X : X \text{ is lower bounded} \} \]
  - Optimal investment in Jin and Zhou has a deterministic loss in a bad market situation.
  - But the loss can be large enough to intrigue disasters, like bankruptcy.
What will we do

- Bankruptcy is not allowed in most market
- Investors may cut loss at some big loss
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- Investors may cut losses at some big loss.
- In our problem,
  - Investors are risk-seeking for losses.
  - Motivate the investor to borrow money for risky investors.
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• Bankruptcy is not allowed in most market

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• In our problem,
  ◦ Investor are risk seeking for loss
  ◦ Motivate the investor to borrow money for risky investor
  ◦ Heavy loss may happen
  ◦ Bankruptcy probability is higher when the investor is more aggressive
What will we do

- Bankruptcy is not allowed in most market
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- In our problem,
  - Investor are risk seeking for loss
  - Motivate the investor to borrow money for risky investor
  - Heavy loss may happen
  - Bankruptcy probability is higher when the investor is more aggressive
- To prevent disaster, a constraint on loss is necessary
Problem with bounded loss

Maximize \( V(X - B) \)

\[
\begin{aligned}
\text{s.t.} & \quad X \geq B - L \\
& \quad E[X \rho] = x_0
\end{aligned}
\]

where \( L \) is an upper bound of loss.
Problem with bounded loss

Maximize \[ V(X - B) \]

s.t. \[
\begin{aligned}
X &\geq B - L \\
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\]

where \( L \) is an upper bound of loss.

Suppose the reference is bounded. Rewrite the problem by changing variable \( \tilde{X} = X - B \),

Maximize \[ V_+(\tilde{X}^+) - V_-(\tilde{X}^-) \]

s.t. \[
\begin{aligned}
\tilde{X} &\geq -L \\
E[\tilde{X}\rho] &= \tilde{x}_0 := x_0 - E[\rho B]
\end{aligned}
\]

where \[ V_\pm(Y) = \int_0^{+\infty} T_\pm(P(u_\pm(y) \geq y))dy. \]
Splitting of the problem

- We use the same splitting from Jin and Zhou (2008)
- For any $c \in (\text{essinf} \rho, \text{esssup} \rho)$, $\tilde{x}_+ \geq \tilde{x}_0^+$, solve the following problems to get their value function $v_\pm(c, \tilde{x}_+)$
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\[
\max \quad V_+ (\tilde{X}_+)
\]

\[
\begin{align*}
\tilde{X}_+ & \geq 0 \\
\tilde{X} & = 0 \text{ when } \rho > c \\
E[\tilde{X}_+ \rho] & = \tilde{x}_+
\end{align*}
\]

(Positive Part Problem)
Splitting of the problem

- We use the same splitting from Jin and Zhou (2008)
- For any $c \in (\inf \rho, \sup \rho)$, $\tilde{x}_+ \geq \tilde{x}_0^+$, solve the following problems to get their value function $v_\pm(c, \tilde{x}_+)$

\[
\begin{align*}
\text{max} \quad & V_+(\tilde{X}_+) \\
\text{s.t.} \quad & \begin{cases} 
\tilde{X}_+ \geq 0 \\
\tilde{X} = 0 \text{ when } \rho > c \\
E[\tilde{X}_+ \rho] = \tilde{x}_+
\end{cases}
\end{align*}
\]

(Positive Part Problem)

\[
\begin{align*}
\text{min} \quad & V_-(\tilde{X}_-) \\
\text{s.t.} \quad & \begin{cases} 
\tilde{X}_- \in [0, L] \\
\tilde{X}_- = 0 \text{ when } \rho < c \\
E[\tilde{X}_- \rho] = \tilde{x}_+ - \tilde{x}_0
\end{cases}
\end{align*}
\]

(Negative Part Problem)
Splitting of the problem

- We use the same splitting from Jin and Zhou (2008)
- For any $c \in (\text{ess inf } \rho, \text{ess sup } \rho)$, $\tilde{x}_+ \geq \tilde{x}_0^+$, solve the following problems to get their value function $v_\pm(c, \tilde{x}_+)$

\[
\begin{align*}
\max & \quad V_+(\tilde{X}_+) \\
\text{s.t.} & \quad \tilde{X}_+ \geq 0 \\
& \quad \tilde{X} = 0 \text{ when } \rho > c \\
& \quad E[\tilde{X}+\rho] = \tilde{x}_+ \\
\text{(Positive Part Problem)}
\end{align*}
\]

\[
\begin{align*}
\min & \quad V_-(-\tilde{X}_-) \\
\text{s.t.} & \quad -\tilde{X}_- \in [0, L] \\
& \quad \tilde{X}_- = 0 \text{ when } \rho < c \\
& \quad E[-\tilde{X}_-\rho] = \tilde{x}_+ - \tilde{x}_0 \\
\text{(Negative Part Problem)}
\end{align*}
\]

- Then find the optimal splitting $c^*$ and $\tilde{x}_+^*$ by solving

\[
\text{Maximize }_{c \in (\text{ess inf } \rho, \text{ess sup } \rho), \tilde{x}_+ \geq \tilde{x}_0^+} v_+(c, \tilde{x}_+) - v_-(c, \tilde{x}_+).
\]
Recovery of optimal contingent claim

- If
  - $c^*, \tilde{x}^*_+$ is an optimal splitting
  - $\tilde{X}^*_+, \tilde{X}^*_-$ are optimal for the two subproblems respectively with parameters $c^*, \tilde{x}^*_+$,

then $X = \tilde{X}^*_+ 1_{\rho \leq c^*} - \tilde{X}^*_- 1_{\rho > c^*} + B$ is optimal
Recovery of optimal contingent claim

- If
  - $c^*, \tilde{x}^+_\ast$ is an optimal splitting
  - $\tilde{X}^+_\ast, \tilde{X}^-_\ast$ are optimal for the two subproblems respectively with parameters $c^*, \tilde{x}^+_\ast$,
  
  then $X = \tilde{X}^+_\ast 1_{\rho \leq c^*} - \tilde{X}^-_\ast 1_{\rho > c^*} + B$ is optimal

- If any of them fails to exist, then there is no optimal contingent claim
Positive part problem solution

The positive part problem is the same as in Jin and Zhou (2008)
Positive part problem solution

- Denote $F_{\rho}(\cdot)$ as the CDF of $\rho$. Suppose it is continuous.

- Suppose (1) $\frac{F_{\rho}^{-1}(\cdot)}{T'_+(\cdot)}$ is $\uparrow$ on $[0, 1]$; (2) $\lim \inf_{x \to +\infty} \frac{-x u'_+(x)}{u'_+(x)} > 0$; (3) $E[u_+((u'_+)^{-1}(\frac{\rho}{T'_+(F_{\rho}(\rho))})) T'_+(F_{\rho}(\rho))] < +\infty$. 
Positive part problem solution

- Denote $F_{\rho}(\cdot)$ as the CDF of $\rho$. Suppose it is continuous.

- Suppose (1) $\frac{F_{\rho}^{-1}(\cdot)}{T'_+(\cdot)}$ is $\uparrow$ on $[0, 1]$; (2) $\liminf_{x \to +\infty} \frac{-xu''_+(x)}{u'_+(x)} > 0$; (3) $E[u_+((u'_+)^{-1}(\frac{\rho}{T'_+(F_{\rho}(\rho))}))(T'_+(F_{\rho}(\rho)))] < +\infty$.

**Theorem 1** For any $c \in (\text{essinf}\, \rho, \text{esssup}\, \rho]$ and $\tilde{x}_+ \geq \tilde{x}_0^+$, the optimal solution for the positive part problem is

$$\tilde{X}_+^* = (u'_+)^{-1}(\lambda \frac{\rho}{T'_+(F_{\rho}(\rho))})1_{\rho \leq c}.$$  

The optimal value is

$$v_+(c, \tilde{x}_+) = E[u_+((u'_+)^{-1}(\lambda \frac{\rho}{T'_+(F_{\rho}(\rho))}))(T'_+(F_{\rho}(\rho))1_{\rho \leq c}),$$

where $\lambda$ is the unique one making $\tilde{X}_+^*$ feasible.
Consider the problem

$$\min_{Y \in [0,L]} E[Y \rho] = a \ V_-(Y)$$
Negative part problem

Consider the problem:

$$\min_{Y \in [0, L], E[Y \rho] = a} V_-(Y)$$

- Notice $V_-(Y)$ only depends on the distribution of $Y$. 
Negative part problem

Consider the problem \[ \min_{Y \in [0, L], E[Y \rho] = a} V_-(Y) \]

- Notice \( V_-(Y) \) only depends on the distribution of \( Y \)
- If \( Y \sim F \), then \( E[Y \rho] \leq E[F^{-1}(F_\rho(\rho))] \)
Negative part problem

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\[
\min_{Y \in [0, L], E[Y \rho] = a} V_-(Y)
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- Notice \( V_-(Y) \) only depends on the distribution of \( Y \).
- If \( Y \sim F \), then \( E[Y \rho] \leq E[F^{-1}(F_{\rho}(\rho))] \).
- \( Y^* \) must be \( Y^* = F^{-1}(F_{\rho}(\rho)) \) with some CDF \( F \).
Consider the problem

$$\min_{Y \in [0, L], E[Y \rho] = a} V_-(Y)$$

- Notice $V_-(Y)$ only depends on the distribution of $Y$
- If $Y \sim F$, then $E[Y \rho] \leq E[F^{-1}(F_\rho(\rho))]$
- $Y^*$ must be $Y^* = F^{-1}(F_\rho(\rho))$ with some CDF $F$
- Denote $Z = F_\rho(\rho)$, $\Gamma = \{F^{-1}(\cdot) : F \text{ is a CDF}\}$ be the set of quantile functions. Then the problem is equivalent to

$$\min \bar{v}_2(g(\cdot)) := E[u_- (g(Z)) T'_-(1 - Z)]$$

s.t. \[\begin{cases} g(\cdot) \in \Gamma, \ g(\cdot) \in [0, L] \text{ on } [0, 1) \\ E[g(Z)F_\rho^{-1}(Z)] = a. \end{cases}\]
Optimal quantile function

- If $g^*(\cdot)$ is optimal quantile function, then $Y^* = g\left(1 - F_{\rho}(\rho)\right)$ is the optimal random variable.
Optimal quantile function

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  - $g^*$ must be on the boundary of the feasible set
  - Without $L$, Jin and Zhou (2008) shows that the boundary consists of $g^*(z; c) := q(c)\mathbf{1}_{z \geq c}$ with proper function $q(\cdot)$ and $c \in (0, 1]$
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  - \( \bar{v}_2(g(\cdot)) \) is concave w.r.t. \( g(\cdot) \)
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  - Without \( L \), Jin and Zhou (2008) shows that the boundary consists of \( g^*(z; c) := q(c) \mathbf{1}_{z \geq c} \) with proper function \( q(\cdot) \) and \( c \in (0, 1] \)
- We need to find out the boundary with the bound \( L \)
Optimal quantile

**Theorem 2** If there are optimal $g(\cdot)$, then one of them is in the form

$$g(x; c_1, c_2) = q(c_1, c_2; a) \mathbf{1}_{x \in [F_\rho(c_1), F_\rho(c_2))] + L \mathbf{1}_{x \geq F_\rho(c_2)},$$

where

$$q(c_1, c_2; a) = \frac{a - L E[\rho \mathbf{1}_{\rho \geq c_2}]}{E[\rho \mathbf{1}_{\rho \in [c_1, c_2)}]}.$$
Optimal quantile

Theorem 2 If there are optimal $g(\cdot)$, then one of them is in the form:

$$g(x; c_1, c_2) = q(c_1, c_2; a)1_{x \in [F_\rho(c_1), F_\rho(c_2))} + L1_{x \geq F_\rho(c_2)},$$

where

$$q(c_1, c_2; a) = \frac{a - LE[\rho 1_{\rho \geq c_2}]}{E[\rho 1_{\rho \in [c_1, c_2)}]}.$$ 

- Only need to solve the problem

$$\min \bar{v}_2(g(\cdot; c_1, c_2))$$

$$s.t. \quad \text{essinf} \rho \leq c_1 < c_2 \leq \text{esssup} \rho$$
Optimal negative part

**Theorem 3** For any $c \in [\text{essinf}\rho, \text{esssup}\rho)$, $\tilde{x}_+ > \tilde{x}_0^+$, the optimal value of the negative part problem is

$$v_-(c, \tilde{x}_+) = \inf_{c \leq c_1 < c_2 \leq \text{esssup}\rho} v_3(c_1, c_2; c, \tilde{x}_+),$$

where

$$v_3(\cdots) = u_-(q(c_1, c_2, \tilde{x}_+ - \tilde{x}_0))(T_-(P(\rho \geq c_2)) - T_-(P(\rho \geq c_1)))$$

$$+ u_-(L)T_-(P(\rho \geq c_2)).$$
Theorem 3 For any \( c \in [\text{essinf}\, \rho, \text{esssup}\, \rho) \), \( \tilde{x}_+ > \tilde{x}_0^+ \), the optimal value of the negative part problem is

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v_-(c, \tilde{x}_+) = \inf_{c \leq c_1 < c_2 \leq \text{esssup}\, \rho) v_3(c_1, c_2; c, \tilde{x}_+),
\]

where

\[
v_3(\cdots) = u_-(q(c_1, c_2, \tilde{x}_+ - \tilde{x}_0))(T_-(P(\rho \geq c_2)) - T_-(P(\rho \geq c_1))) + u_-(L)T_-(P(\rho \geq c_2)).
\]

Furthermore, if \( v_-(c, x_+) \) is obtained at \((c_1^*, c_2^*)\), then

\[
\tilde{X}_-^* = q(c_1^*, c_2^*; \tilde{x}_+^* - \tilde{x}_0)1_{\rho \in [c_1^*, c_2^*)} + L1_{\rho \geq c_2^*}
\]

is an optimal solution for the negative part problem.
The optimal splitting $c^*, \tilde{x}^*_+$ can be determined by

$$\max \ v_+(c, \tilde{x}_+) - v_3(c, c_2; c, \tilde{x}_+)$$

$$s.t. \ \tilde{x}_+ \geq \tilde{x}_0, \essinf \rho \leq c < c_2 \leq \esssup \rho$$
Optimal terminal wealth

The optimal splitting \( c^*, \tilde{x}^* \) can be determined by

\[
\max \; v_+(c, \tilde{x}_+) - v_3(c, c_2; c, \tilde{x}_+) \\
\text{s.t.} \; \tilde{x}_+ \geq \tilde{x}_0, \; \text{essinf} \rho \leq c < c_2 \leq \text{esssup} \rho
\]

**Theorem 4** Under the assumption made for positive part problem,

(i) If \((c^*, c_2^*, \tilde{x}_+^*)\) is an optimal splitting, then

\[
X^* = (u_+^*)^{-1}(\lambda \frac{\rho}{T'_+(F(\rho))})1_{\rho \leq c^*} - q(c^*, c_2^*; \tilde{x}_+^* - \tilde{x}_0)1_{\rho \in [c^*, c_2^*)} - L1_{\rho \geq c_2^*} + B
\]

is an optimal terminal wealth.
Optimal terminal wealth

The optimal splitting $c^*, \tilde{x}_+^*$ can be determined by

$$\max v_+(c, \tilde{x}_+) - v_3(c, c_2; c, \tilde{x}_+)$$

$$s.t. \quad \tilde{x}_+ \geq \tilde{x}_0, \text{essinf} \rho \leq c < c_2 \leq \text{esssup} \rho$$

**Theorem 4** Under the assumption made for positive part problem,

(i) If $(c^*, c_2^*, \tilde{x}_+^*)$ is an optimal splitting, then

$$X^* = (u'_+)^{-1}(\lambda \frac{\rho}{T'_+(F(\rho))})1_{\rho \leq c^*} - q(c^*, c_2^*; \tilde{x}_+^* - \tilde{x}_0)1_{\rho \in [c^*, c_2^*)} - L1_{\rho \geq c_2^*} + B$$

is an optimal terminal wealth.

(ii) If there is no optimal $(c, c_2, \tilde{x}_+)$, then there is no optimal terminal wealth.
Example: power value function

- Generally, $X^*$ is a three-piece function of $\rho$
Example: power value function

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• Consider the example with $u_+(x) = x^\alpha$, $u_-(x) = kx^\alpha$ for some $k > 1$ and $\alpha \in (0, 1)$
  
  ◦ In this example, optimal solution always exists
Example: power value function

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• Consider the example with $u_+(x) = x^\alpha$, $u_-(x) = k x^\alpha$ for some $k > 1$ and $\alpha \in (0, 1)$
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• Define $f_1 = 1 - F_\rho$, $f_2(x) = E[\rho 1_{\rho \geq x}]$, $f(x) = f_2(f_1^{-1}(x))$
Example: power value function

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- Consider the example with $u_+(x) = x^\alpha$, $u_-(x) = kx^\alpha$ for some $k > 1$ and $\alpha \in (0, 1)$
  - In this example, optimal solution always exists

- Define $f_1 = 1 - F_\rho$, $f_2(x) = E[\rho 1_{\rho \geq x}]$, $f(x) = f_2(f_1^{-1}(x))$

**Theorem 5** If $h(x) = T_-(f^{-1}(x)))$ is a convex function, then the optimal splitting $(c^*, c_2^*, x_+^*)$ satisfies $c^* = c_2^*$. Hence the optimal contingent claim is

$$X^* = (u'_+)^{-1}(\lambda \frac{\rho}{T'_+(F(\rho))}) 1_{\rho \leq c_2^*} - L 1_{\rho \geq c_2^*} + B.$$
Example: power value function

• Consider the case $h(x) = x^\beta$ with $\beta > 0$

• If $\beta < 1$, Theorem 5 does not apply
Example: power value function

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- If $\beta < 1$, Theorem 5 does not apply

**Theorem 6** Given $h(x) = x^\beta$ for some $\beta > 0$. Then

- If $\beta \geq \alpha$, then $c^*_2 = c^*$, and

  $$X^* = (u'_+)^{-1} \left( \lambda \frac{\rho}{T'_{(F(\rho))}} 1_{\rho \leq c^*_2} - L 1_{\rho \geq c^*_2} + B. \right)$$
Example: power value function

- Consider the case \( h(x) = x^\beta \) with \( \beta > 0 \)

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**Theorem 6**

Given \( h(x) = x^\beta \) for some \( \beta > 0 \). Then

- If \( \beta \geq \alpha \), then \( c_2^* = c^* \), and
  \[
  X^* = (u'_+)^{-1}(\lambda_{T_+}(F(\rho)))1_{\rho \leq c_2^*} - L1_{\rho \geq c_2^*} + B.
  \]

- If \( \beta < \alpha \), then \( c_2^* = +\infty \), and
  \[
  X^* = (u'_+)^{-1}(\lambda_{T_+}(F(\rho)))1_{\rho \leq c^*} - \frac{\tilde{x}_+ - \tilde{x}_0}{E\rho 1_{\rho \geq c^*}}1_{\rho \geq c^*} + B.
  \]
Example: power value function

- Consider the case \( h(x) = x^\beta \) with \( \beta > 0 \)
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**Theorem 6** Given \( h(x) = x^\beta \) for some \( \beta > 0 \). Then

- If \( \beta \geq \alpha \), then \( c_2^* = c^* \), and
  \[ X^* = (u'_+)^{-1}(\lambda\frac{\rho}{T'_+(F(\rho))})1_{\rho \leq c_2^*} - L1_{\rho \geq c_2^*} + B. \]
- If \( \beta < \alpha \), then \( c_2^* = +\infty \), and
  \[ X^* = (u'_+)^{-1}(\lambda\frac{\rho}{T'_+(F(\rho))})1_{\rho \leq c^*} - \frac{\tilde{x}^*_+ - \tilde{x}_0}{E\rho1_{\rho \geq c^*}}1_{\rho \geq c^*} + B. \]

In any case, \( X^* \) is a two-piece function of \( \rho \).
Example: power value function

- Is the optimal solution always two-piece for power value function?
Example: power value function

- Is the optimal solution always two-piece for power value function?

- A three-piece example:
  
  - $L = 10$, $\tilde{x}_0 = -1$, $\beta = 0.85$, $\alpha = 0.88$, $k = 2.25$,
  
  $\rho \sim \text{Lognormal}(-0.045, 0.09)$
  
  - $h(x) = \begin{cases} 
  0.5x & x \in [0, 0.05] \\
  20 \cdot 0.1^\beta (x - 0.05) + 0.025(0.1 - x) & x \in [0.05, 0.1] \\
  x^\beta & x \in [0.1, 1] \end{cases}$

- The optimal solution $\tilde{X}^* = X^* - B$ is as in the next figure
Example: power value function
Thank you very much!