Duality for set-valued measures of risk

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With: F. Heyde (Halle), B. Rudloff & M. Yankova (Princeton)
Basic question.

How to evaluate the risk of $X \in L^0_d = L^0(\Omega, \mathcal{F}, P; \mathbb{R}^d)$?
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How to evaluate the risk of $X \in L_0^d = L^0(\Omega, \mathcal{F}, P; \mathbb{R}^d)$?

Basic problems.

(1) $u^1, u^2 \in \mathbb{R}^d$ compensate for the risk of $X$, but might not be comparable.
Basic question.
How to evaluate the risk of $X \in L^0_d = L^0(\Omega, \mathcal{F}, P; \mathbb{R}^d)$?

Basic problems.

(1) $u^1, u^2 \in \mathbb{R}^d$ compensate for the risk of $X$, but might not be comparable.

Example. 1-1 exchange rate, 10% transaction costs: neither of

$$u^1 = \begin{pmatrix} 1000 \\ 0 \end{pmatrix}, \quad u^2 = \begin{pmatrix} 0 \\ 1000 \end{pmatrix}$$

is "better".
Basic question.

How to evaluate the risk of $X \in L^0_d = L^0(\Omega, \mathcal{F}, P; \mathbb{R}^d)$?

Basic problems.

(1) $u^1, u^2 \in \mathbb{R}^d$ compensate for the risk of $X$, but might not be comparable.

(2) $u^1 \in \mathbb{R}^d$ does not compensate for the risk of $X$, but can be exchanged at initial time into $u^2 \in \mathbb{R}^d$ which does.

(3) $u \in \mathbb{R}^d$ does not compensate for the risk of $X^1$, but $X^1$ can be exchanged at terminal time into $X^2$ such that $u$ compensates for $X^2$. 
Basic idea.

$A \subseteq L^0_d$ set of acceptable payoffs: The mapping

$$X \mapsto R_A(X) = \{ u \in \mathbb{R}^d : X + u \mathbb{I} \in A \} \subseteq \mathcal{P}(\mathbb{R}^d)$$

is understood as a set-valued risk measure $R_A : L^0_d \to \mathcal{P}(\mathbb{R}^d)$. 

References.

Superhedging theorems for markets with transaction costs: Kabanov 99, Schachermayer 04, Pennanen/Penner 10...

Set-valued risk measure ad hoc: Jouini/Touzi/Meddeb 04

Complete theory, constant cone: Hamel/Heyde 10

Complete theory, random cone: Hamel/Heyde/Rudloff 10+
Basic idea.

$A \subseteq L^0_d$ set of acceptable payoffs: The mapping

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References.

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Complete theory, constant cone: Hamel/Heyde 10
Complete theory, random cone: Hamel/Heyde/Rudloff 10+
Rest of the talk.

• Formal definitions and primal representation
• Dual representation and dual variables
• Super-hedging price as a coherent SRM
• A set-valued AV@R: definition and computation
Formal definitions.

Space of eligible portfolios.

- $M \subseteq \mathbb{R}^d$ linear subspace, e.g. $M = \mathbb{R}^m \times \{0\}^{d-m}$
Formal definitions.

Space of eligible portfolios.

- $M \subseteq \mathbb{R}^d$ linear subspace, e.g. $M = \mathbb{R}^m \times \{0\}^{d-m}$

Acceptance sets. $A \subseteq L^p_d$, $0 \leq p \leq \infty$, with

(A1) $M \mathbb{1} \cap A \neq \emptyset$, $M \mathbb{1} \cap (L^p_d \setminus A) \neq \emptyset$

(A2) $A + (L^p_d)_+ \subseteq A$. 

Note. Set-valuedness solves the problem of incomparableness!
Formal definitions.

**Space of eligible portfolios.**

- $M \subseteq \mathbb{R}^d$ linear subspace, e.g. $M = \mathbb{R}^m \times \{0\}^{d-m}$

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**Risk measures.** $R_A : L^p_d \rightarrow \mathcal{P}(M)$ defined by

$$R_A(X) = \{u \in M : X + u \mathbb{1} \in A\}, \quad X \in L^p_d.$$

**Note.** Set-valuedness solves the problem of incomparableness!
Result. The set-valued function $X \mapsto R_A(X)$ is

(R0) $M$-translative, i.e.

$$\forall X \in L^p_d, \forall u \in M: R(X + u \mathbb{1}) = R(X) - u.$$ 

(R1) finite at zero: $R(0) \neq \emptyset$ and $R(0) \neq M$.

(R2) $(L^p_d)_+$-monotone, i.e.

$$X^2 - X^1 \in (L^p_d)_+ \implies R(X^2) \supseteq R(X^1).$$
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$$X^2 - X^1 \in (L^p_d)_+ \Rightarrow R(X^2) \supseteq R(X^1).$$

$M$-translative functions and some subsets of $L^p_d$ are one–to–one via

$$A_R = \left\{ X \in L^p_d : 0 \in R(X) \right\}, \quad R_A(X) = \left\{ u \in M : X + u \mathbb{1} \in A \right\}$$
Conical market models with one period.

At **Initial Time**.
- \( K_I \subseteq \mathbb{R}^d \) a solvency cone: closed convex cone with \( \mathbb{R}_+^d \subseteq K_I \neq \mathbb{R}^d \)
- \( K_I^M = K_I \cap M \) solvency cone restricted to eligible portfolios

**\( K_I \)-compatible:** \( X \in A, u \in K_I^M \Rightarrow X + u I \in A \).
Conical market models with one period.

At **Initial Time.**

- $K_I \subseteq \mathbb{R}^d$ a solvency cone: closed convex cone with $\mathbb{R}_+^d \subseteq K_I \neq \mathbb{R}^d$
- $K^M_I = K_I \cap M$ solvency cone restricted to eligible portfolios

  $K_I$-compatible: $X \in A, u \in K^M_I \Rightarrow X + u\mathbb{1} \in A.$

At **Terminal Time.**

- $K_T: \Omega \rightarrow \mathcal{P}(\mathbb{R}^d)$ (measurable) solvency cone mapping

  $K_T$-compatible: $X \in A, X' \in K_T$ a.s. $\Rightarrow X + X' \in A.$
One-to-one properties for $M$-translative functions $R$ and $A \subseteq L^p_d$:

$$A_R = \left\{ X \in L^p_d : 0 \in R(X) \right\}, \quad R_A(X) = \left\{ u \in M : X + u \mathbb{I} \in A \right\}$$

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<td>$R(0) \neq M$</td>
<td>$M \mathbb{I} \cap (L^p_d \setminus A) \neq \emptyset$</td>
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<td>$R(X) = R(X) + K^M_0$</td>
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Duality.

**Result.** If a function $R : L^p_d \rightarrow \mathcal{P}(M)$ is convex (closed), then $R(X)$ is convex (closed) for all $X \in L^p_d$. A closed convex $K_I$-compatible risk measure $R$ maps into

$$G(M) = \left\{ D \subseteq \mathbb{R}^d : D = \text{cl} \text{co} \left( D + K^M_I \right) \right\}.$$

Here: convexity, closedness in terms of the graph

$$\text{gr} \ R = \left\{ (X, u) \in L^p_d \times M : u \in R(X) \right\}.$$
Dual representation theorem. $R: L^p_d \rightarrow \mathcal{G}(M)$ is a closed convex market-compatible risk measure if and only if there is a penalty function $-\alpha: \mathcal{W}^q \rightarrow \mathcal{G}(M)$ such that for all $X \in L^p_d$

$$R(X) = \bigcap_{(Q, w) \in \mathcal{W}^q} \left\{ -\alpha(Q, w) + \left( E^Q[-X] + G(w) \right) \cap M \right\}.$$
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$$R(X) = \bigcap_{(Q,w) \in \mathcal{W}^q} \left\{ -\alpha(Q,w) + \left( E^Q[-X] + G(w) \right) \cap M \right\}.$$ 

In this case,

$$-\alpha(Q,w) \subseteq \text{cl} \bigcup_{X' \in A_R} \left( E^Q[X'] + G(w) \right) \cap M$$

with $G(w) = \left\{ x \in \mathbb{R}^d : 0 \leq w^T x \right\}$ and

$$\mathcal{W}^q = \left\{ (Q,w) \in \mathcal{M}^P_{1,d} \times \left( K^+_I \setminus M^\perp + M^\perp \right) : \text{diag} (w) \frac{dQ}{dP} \in L^q_d \left( K^+_T \right) \right\}.$$
A note about the proof. Fenchel-Moreau theorem for set-valued functions, Hamel 09.
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A note about dual variables. Assume $M = \mathbb{R}^d$. Then

$$\mathcal{W}^q = \left\{ (Q, w) \in \mathcal{M}^{P}_{1,d} \times K^+_I \setminus \{0\} : \text{diag}(w) \frac{dQ}{dP} \in L^q_d\left(K^+_T\right) \right\}.$$
A note about the proof. Fenchel-Moreau theorem for set-valued functions, Hamel 09.

A note about dual variables. Assume $M = \mathbb{R}^d$. Then

$$W^q = \left\{ (Q, w) \in \mathcal{M}_{1,d}^P \times K_I^+ \setminus \{0\} : \text{diag}(w) \frac{dQ}{dP} \in L_q^d(K_T^+) \right\}. $$

Transformation of variables. $Y = \text{diag}(w) \frac{dQ}{dP}$, $E[Y] = w \in K_I^+ \setminus \{0\}$.

This gives: The pair $(Y, w)$ is a consistent pricing process for the one-period market $(K_I, K_T = K_T(\omega))$. 
The coherent case. $R$ additionally positively homogeneous:

$$\forall X \in L^p_d: R(X) = \bigcap_{(Q,w)\in \mathcal{W}^q_R} \left( E^Q [-X] + G(w) \right) \cap M.$$ 

with

$$\mathcal{W}^q_R \subseteq \left\{ (Q, w) \in \mathcal{M}^P_{1,d} \times \left( K^+_I \setminus M^\perp + M^\perp \right) : \text{diag} \,(w) \frac{dQ}{dP} \in A^+_R \right\}.$$
The coherent case. \( R \) additionally positively homogeneous:

\[
\forall X \in L^p_d : R(X) = \bigcap_{(Q,w) \in \mathcal{W}^q_R} \left( E^Q [-X] + G(w) \right) \cap M.
\]

with

\[
\mathcal{W}^q_R \subseteq \left\{ (Q, w) \in M^P_{1,d} \times \left( K_I^+ \setminus M^\perp + M^\perp \right) : \text{diag}(w) \frac{dQ}{dP} \in A^+_R \right\}.
\]

The coherent case with \( M = \mathbb{R}^d \).

\[
\mathcal{W}^q_R \subseteq \left\{ (Q, w) \in M^P_{1,d} \times K_I^+ \setminus \{0\} : \text{diag}(w) \frac{dQ}{dP} \in A^+_R \right\}.
\]
Super-hedging price.

- $\Theta = \{t_0 = 0, t_1, \ldots, t_N = T\}$, $(\Omega, (\mathcal{F}_t)_{t \in \Theta}, \mathcal{F}, P)$, $\mathcal{F}_T = \mathcal{F}$;

- $(K_t(\omega))_{t \in \Theta}$ cone-valued process with $\mathbb{R}_+^d \subseteq K_t(\omega) \subseteq \mathbb{R}^d$, $K_t(\omega) \neq \mathbb{R}^d$ closed convex $P$-a.s. for all $t \in \Theta$;

- **Self-financing portfolio process:** adapted $\mathbb{R}^d$–valued process $V = (V_t)_{t \in \Theta}$ with $(V_{t-1} = 0)$

  $$V_{tn} - V_{tn-1} \in -K_{tn} \quad \text{a.s., } n = 1, \ldots, N - 1$$

- The **attainable set**

  $$A_t = \{V_t: V \text{ is a self-financing portfolio process}\}, \ t \in \Theta$$

  is a convex cone in $L^0\left(\Omega, \mathcal{F}_t, P; \mathbb{R}^d\right)$.
**Result.** Assume \((\text{NA}^r)\). Then \(X \mapsto \{ u \in \mathbb{R}^d : X + u\mathbb{1} \in -A_T \} \) is a closed coherent market-compatible risk measure with \(K_I = K_0\).

**Note.** \(-A_T = K_0\mathbb{1} + L^0_d(K_{t_1}) + \ldots + L^0_d(K_T)\).
**Result.** Assume (NA\(^r\)). Then \( X \mapsto \{ u \in \mathbb{R}^d : X + u \mathbb{1} \in -A_T \} \) is a closed coherent market-compatible risk measure with \( K_I = K_0 \).

**Note.** \(-A_T = K_0 \mathbb{1} + L_d^0(K_{t_1}) + \ldots + L_d^0(K_T)\).

**Super-hedging theorem.** \( X \in L^1_d, \ v \in \mathbb{R}^d \)

\[ X - v \mathbb{1} \in A_T \quad \iff \quad \forall Z \in SCPP: \ E[X^T Z_T] \leq v^T Z_0. \]

This produces the dual representation of the super-hedging price in terms of \((Q, w)\) via the following transformation of variables.
Transformation of variables. Set $w = E[Z_T] = Z_0 \in K_0^+ \setminus \{0\}$ and

$$
\frac{dQ_i}{dP} = \frac{1}{w_i} (Z_T)_i \quad \text{if } w_i > 0,
$$

and choose $\frac{dQ_i}{dP}$ as density in $L_\infty^+$ if $w_i = 0$. Then

$$(Q, w) \in \mathcal{M}_{1,d}^P \times K_0^+ \setminus \{0\}$$

$$E \left[ \text{diag} \left( w \right) \frac{dQ}{dP} | \mathcal{F}_t \right] \in L_d^p(K_t^+), \ t \in \Theta$$

In particular, $\text{diag} \left( w \right) \frac{dQ}{dP} \in K_T^+ P$-a.s. Moreover, $E \left[ X^T Z_T \right] = w^T E^Q [X]$ and $Z_0^T u = w^T u$, hence the following result.
**Result.** \( X \in L^1_d \). Then,

\[
R_{-A_T} (-X) = \bigcap_{(Q, w) \in \mathcal{W}^\infty_{SCPP}} \left( E^Q [X] + G(w) \right)
\]

with

\[
\mathcal{W}^\infty_{SCPP} = \left\{ (Q, w) \in \mathcal{M}^P_{1,d} \times K^+_0 \setminus \{0\} : \forall t \in \Theta: E \left[ \text{diag} (w) \frac{dQ}{dP} \big| \mathcal{F}_t \right] \in L^p_d \left(K_t^+ \right) \right\}.
\]

**Summary.** Set-valued duality covers both super-hedging theorems and dual representation of risk measures in conical market models.
Recall (from dual representation theorem for $q = \infty$)
\[
\mathcal{W}^\infty = \left\{ (Q, w) \in \mathcal{M}_{1,d}^P \times \left( K_I^+ \backslash M^\perp + M^\perp \right) : \text{diag} (w) \frac{dQ}{dP} \in L^\infty_1 \left( K_T^+ \right) \right\}.
\]

If $\alpha \in (0, 1]^d$,
\[
\mathcal{W}_\alpha^\infty = \left\{ (Q, w) \in \mathcal{W}^\infty : \text{diag} (w) \left( \alpha \mathbb{1} - \frac{dQ}{dP} \right) \in L^\infty_1 \left( K_T^+ \right) \right\}
\]
then
\[
AV@R\alpha (X) = \bigcap_{(Q, w) \in \mathcal{W}_\alpha^\infty} \left( \mathbb{E}_Q [-X] + G (w) \right) \cap M
\]
defines a market-compatible sublinear (coherent) risk measure on $L^1_d$.

Note. This is a "dual-way" definition! And a new one, by the way.
Questions.

1. Computing values $AV@R_\alpha(X)$?

2. Minimizing $AV@R_\alpha(X)$ over $X \in C \subseteq L^1_d$?
Computing the value $AV \otimes R_\alpha(X)$.

**Fact 1.**

$$AV \otimes R_\alpha(X) = \bigcap_{(Q,w) \in \mathcal{W}_\alpha} \left( E^Q [-X] + G(w) \right) \cap M$$

$$= \bigcap_{(Y,v) \in \mathcal{Y}_\alpha} \left\{ u \in M : E \left[ -Y^T X \right] \leq v^T u \right\}$$

with

$$\mathcal{Y}_\alpha = \left\{ (Y,v) \in L^\infty_{d} \times M \setminus \{0\} : \begin{array}{l} v \in (E[Y] + M^\perp) \cap (K^+_I + M^\perp) \vspace{1em} \text{and} \vspace{1em} Y \in K^+_T \setminus \{0\} \vspace{1em} \text{with} \vspace{1em} \text{diag} (\alpha) E[Y] - Y \in K^+_T \end{array} \right\}.$$ 

**Note.** Linear in $(Y,v)$. 
Computing the value $AV @ R_\alpha (X)$. 

**Fact 2.** If $M = \mathbb{R}^d$ this simplifies to 

$$AV @ R_\alpha (X) = \bigcap_{(Q, w) \in W^\infty_\alpha} \left( E^Q [-X] + G (w) \right)$$

$$= \bigcap_{(Y, v) \in \mathcal{Y}^d_\alpha} \left\{ u \in \mathbb{R}^d : E [-Y^T X] \leq v^T u \right\}$$

with 

$$\mathcal{Y}^d_\alpha = \left\{ (Y, v) \in L^\infty_d \left( K_T^+ \right) \times K_I^+ \setminus \{0\} : \right.$$ 

$$v = E [Y], \ \text{diag} (\alpha) v - Y \in L^\infty_d \left( K_T^+ \right) \right\}.$$
Computing the value $AV@R_{\alpha}(X)$.

Further assumptions.

- $|\Omega|, M = \mathbb{R}^d$
- $K_I$ is spanned by $h^1, \ldots, h^J_I$
- $K_T(\omega)$ is spanned by $k^1(\omega), \ldots, k^J_T(\omega)$

Note.

- $Y \in K_T^+ \iff Y \geq 0$
- $\text{diag}(\alpha)v - Y \in K_T^+ \text{ P-a.s.} \iff Y \leq \text{diag}(\alpha)v$
- $\bigcap \iff \sup$
- $X \mapsto \{u \in \mathbb{R}^d : E[-Y^TX] \leq v^Tu\} "\text{almost linear}"$
Computing the value $AV@R_\alpha(X)$.

Analyzing the constraints.

- $Y \in K_T^+$: $y_{in} = Y_i(\omega_n), \ i = 1, \ldots, d, \ n = 1, \ldots, N$

  $\forall j = 1, \ldots, J_T, \forall n = 1, \ldots, N: \sum_{i=1}^{d} y_{in}k_{in}^j \geq 0$

  with $k_{in}^j = k_i^j(\omega_n)$. This gives $NJ_T$ linear inequality constraints.
Computing the value $AV@R_\alpha(X)$.

Analyzing the constraints.

- $\text{diag}(\alpha)v - Y \in K_T^+$:

  $$\forall j = 1, \ldots, J_T, \forall n = 1, \ldots, N: \sum_{i=1}^{d} y_{in}k_{in}^j \leq \sum_{i=1}^{d} \alpha_i k_{in}^j v_i.$$ 

  This gives another $NJ_T$ linear inequality constraints.
Computing the value $AV \otimes R_\alpha(X)$.

Analyzing the constraints.

- $\text{diag}(\alpha)v - Y \in K_T^+$:

  $$\forall j = 1, \ldots, J_T, \forall n = 1, \ldots, N: \sum_{i=1}^{d} y_{in}k_{in}^j \leq \sum_{i=1}^{d} \alpha_i k_{in}^j v_i.$$  

  This gives another $NJ_T$ linear inequality constraints.

- $v = E[Y]$:

  $$\forall i = 1, \ldots, d: \sum_{n=1}^{N} p_n y_{in} = v_i.$$  

  This gives $d$ linear equations.
Computing the value $AV@R_\alpha(X)$.

Analyzing the objective.

- $\{u \in \mathbb{R}^d: E[-Y^T X] \leq v^T u\}$:

  $$E[-Y^T X] = - \sum_{i=1}^{d} \sum_{n=1}^{N} p_n x_{in} y_{in},$$

  therefore the objective becomes

  $$S(\hat{D} \hat{y}, -v)(-\hat{x}) = \{u \in \mathbb{R}^d: -\hat{x}^T \hat{D} \hat{y} \leq v^T u\}.$$
Computing the value $AV@R_\alpha (X)$.

Analyzing the objective.
• $\{u \in \mathbb{R}^d: E \left[ -Y^T X \right] \leq v^T u \}$:

$$E \left[ -Y^T X \right] = - \sum_{i=1}^{d} \sum_{n=1}^{N} p_n x_{in} y_{in},$$

therefore the objective becomes

$$S(\hat{D}\hat{y}, -v) (-\hat{x}) = \{u \in \mathbb{R}^d: - \hat{x}^T \hat{D}\hat{y} \leq v^T u \}.$$

Altogether.

$AV@R_\alpha (X) = \bigcap \left\{ S(\hat{D}\hat{y}, -v) (-\hat{x}) : A_1^T \hat{y} \leq -C_1^T v, A_2^T \hat{y} = -C_2^T v, v \in K_I^+ \right\}$

with suitable matrices $A_1, A_2, C_1, C_2, \hat{D}, \hat{x}, \hat{y}$.

Reference. Yankova 10, JP, P.U.
Computing the value $AV@R_\alpha (X)$.

Constructing the primal.

The problem

$$\bigcap \left\{ S(\bar{D}\hat{y},-v) (-\hat{x}) : A_1^T\hat{y} \leq -C_1^Tv, A_2^T\hat{y} = -C_2^Tv, v \in K_I^+ \right\}$$

is the set-valued dual of the following set-valued linear program

$$\inf_{G(\mathbb{R}^d)} \left\{ C_1x^1 + C_2x^2 : A_1x^1 + A_2x^2 = -\hat{x}, x^1 \geq 0 \right\}.$$
Computing the value $AV_{R^\alpha} (X)$.

Constructing the primal.

The problem

$$\bigcap \left\{ S(D\hat{y}, -v) (-\hat{x}) : A_1^T \hat{y} \leq -C_1^T v, \ A_2^T \hat{y} = -C_2^T v, \ v \in K_I^+ \right\}$$

is the set-valued dual of the following set-valued linear program

$$\inf_{G(\mathbb{R}^d)} \left\{ C_1 x_1 + C_2 x_2 : A_1 x_1 + A_2 x_2 = -\hat{x}, \ x_1 \geq 0 \right\}.$$ 

Interpretation as vector optimization problem. Look for minimal points of

$$\left\{ \text{diag} (\alpha) E [Z] - z : Z \in L_q^q (K_T), \ Z - z \mathbb{I} + X \in L_q^q (K_T), \ z \in \mathbb{R}^d \right\}$$

with respect to the order relation in $\mathbb{R}^d$ generated by $K_I$.

Reference. Hamel 10+
Computing the value $AV@R_\alpha (X)$.

Under the additional assumptions and $M = \mathbb{R}^d$

$AV@R_\alpha (X)$

$= \{ \text{diag} (\alpha) E[Z] - z : Z \in L^q_d (K_T), \ Z - zI + X \in L^q_d (K_T), \ z \in \mathbb{R}^d \}$

$= \bigcap_{(Y,v) \in \mathcal{Y}_\alpha^d} \{ u \in \mathbb{R}^d : E[-YT X] \leq v^T u \}$

with

$\mathcal{Y}_\alpha^d = \left\{ (Y,v) \in L^\infty_d \left( K_T^+ \right) \times K_T^+ \setminus \{0\} : v = E[Y], \ \text{diag} (\alpha) v - Y \in K_T^+ \right\}$
Computing the value $AV@R_\alpha(X)$.

Under the additional assumptions and $M = \mathbb{R}^d$

$$AV@R_\alpha(X)$$

$$= \{ \text{diag}(\alpha) E[Z] - z : Z \in L^q_d(K_T), \ Z - z I + X \in L^q_d(K_T), \ z \in \mathbb{R}^d \}$$

$$= \bigcap_{(Y,v) \in \mathcal{Y}_\alpha^d} \{ u \in \mathbb{R}^d : E[-YT X] \leq v^T u \}$$

with

$$\mathcal{Y}_\alpha^d = \left\{ (Y,v) \in L^\infty_d \left( K^+_T \right) \times K^+_I \setminus \{0\} : v = E[Y], \ \text{diag}(\alpha) v - Y \in K^+_T \right\}$$

Good news. There are already efficient algorithms for such (vector) problems (Benson 1998, Ehrgott/Löhne/Shao 2007).
Computing the value $AV@R_\alpha(X)$.

Under the additional assumptions and $M = \mathbb{R}^d$

$$AV@R_\alpha(X) = \{ \text{diag}(\alpha) E[Z] - z : Z \in L^q_d(K_T), Z - zI + X \in L^q_d(K_T), z \in \mathbb{R}^d \}$$

$$= \bigcap_{(Y,v) \in \mathcal{Y}_\alpha^d} \{ u \in \mathbb{R}^d : E[-YT X] \leq v^T u \}$$

with

$$\mathcal{Y}_\alpha^d = \left\{ (Y, v) \in L^\infty_d \left(K_T^+\right) \times K_T^+ \setminus \{0\} : v = E[Y], \text{diag}(\alpha)v - Y \in K_T^+ \right\}$$

**Good news.** There are already efficient algorithms for such (vector) problems (Benson 1998, Ehrgott/Löhne/Shao 2007).

**Summary.** Computation of values of a set-valued risk measure is a vector/set optimization problem. Set-valued duality provides tools.
What’s next?

- Computing super-hedging prices and values of AV@R.
- Set-valued optimization problems for set-valued risk measures.
- Law invariance of set-valued risk measures.
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Thanks for coming.