When can a risk measure be updated consistently?

Berend Roorda\textsuperscript{1} Hans Schumacher\textsuperscript{2}

\textsuperscript{1}FELab and School of Management and Governance
University of Twente, the Netherlands

\textsuperscript{2}CentER and Department of Econometrics and Operations Research
Tilburg University, the Netherlands

Research sponsored in part by Netspar

BFS Toronto June 23 2010
Outline

- Intro to updating
- Weak time consistency
- Unique updating
- Concrete: extra power in weakly consistent entropic risk measures
Updating an expectation operator

\[ E^Q X \]

**Law of iterated expectations:**

\[ E_Q X = E_Q [E_Q X | F_t] \in L_\infty(\Omega, F, P), Q \ll P \]

\[ 0 \quad t \]
Updating an expectation operator

0

$E^Q X$

$t$

$E^Q_t X$
Updating an expectation operator

\[ 0 \quad t \]

\[ E^Q X \quad E_t^Q X \]

Law of iterated expectations:

\[ E^Q X = E^Q E_t^Q X \]
Updating an expectation operator

\[ E^Q X = E^Q E_t^Q X \]

Law of iterated expectations:

\[ \begin{array}{ccc}
0 & \rightarrow & t \\
E^Q X & \rightarrow & E_t^Q X \\
\end{array} \]

\[ X \in L^\infty(\Omega, \mathcal{F}, P), \ Q \ll P \]

\[ E_t^Q X \text{ also often written as } E^Q[X|\mathcal{F}_t] \]
Updating a coherent risk measure

\[ \phi_0(X) = \inf_{Q \in \mathcal{Q}} E^Q X \]
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\[ \phi_0(X) = \inf_{Q \in \mathcal{Q}} E^Q X \quad \phi_t(X) = \inf_{Q \in \mathcal{Q}} E^Q_t X \]
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Strongly time consistent:

\[ \phi_0(X) = \phi_0(\phi_t(X)) \]
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Strongly time consistent:

\[ \phi_0(X) = \phi_0(\phi_t(X)) \text{ iff } Q \text{ has pasting property} \]

Delbean 2003
extension to convex class Föllmer and Penner 2006
Strong time consistency???

\[ \phi_0(X) = \phi_0(\phi_t(X)) \]

What if the risk measure resembles a capital charge?
Strong time consistency???

$$\phi_0(X) = \phi_0(\phi_t(X))$$

What if the risk measure resembles a capital charge?

Strong time consistency requires that (at time 0) you don’t discriminate between the depicted payoff distribution (in some state, at time t, say) and its risk level $\phi_t(X)$ indicated by the dot . . .
Strong time consistency???

\[ \phi_0(X) = \phi_0(\phi_t(X)) \]

What if the risk measure resembles a capital charge?

Strong time consistency requires that (at time 0) you don’t discriminate between the depicted payoff distribution (in some state, at time t, say) and its risk level \( \phi_t(X) \) indicated by the dot . . .

*Strong time consistency is inappropriate for risk measures that (which?) are much more conservative than pricing measures*
Sequential consistency is the combination of

\[
\begin{align*}
\text{Acceptance consistency:} & \quad \phi_s(X) \geq 0 \iff \phi_t(X) \geq 0 \\
\text{Rejection consistency:} & \quad \phi_s(X) \leq 0 \iff \phi_t(X) \leq 0
\end{align*}
\]

Intuition: capital charges should not increase/decrease with probability 1

Necessity quite convincing for risk

Indeed weaker than strong (under sensitivity assumption)

Sufficient for unique updates in the entire class of conditional evaluations (=normalized monetary risk measures)

Weak time consistency

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The refinement update

$\phi_0$

$\Phi$

$\mathcal{A}$

Detlefsen & Scandolo 2005; Cheridito Delbean Kupper 2006

$\phi_t$ is the only candidate for a weakly time consistent update and can only be consistent if equality holds ("conditional consistency", $\mathcal{A}_t = \mathcal{A}_t$)
The refinement update

\[ \phi_0 \]

\[ \iff \]

\[ \mathcal{A} \to \mathcal{A}^t \]

\[ \{ X \mid \forall F \in F_t : 1_F X \in \mathcal{A} \} \]
The refinement update

\[ \phi_0 \]

\[ \uparrow \uparrow \]

\[ \mathcal{A} \rightarrow \mathcal{A}^t \subseteq \mathcal{A}_t \]

\[ \{ X \mid \forall F \in \mathcal{F}_t : 1_F X \in \mathcal{A} \} \]
The refinement update

\[ \phi_0 \quad \iff \quad \phi_t \]

\[ \mathcal{A} \quad \xrightarrow{\text{consistent update}} \quad \mathcal{A}^t \quad \subseteq \quad \mathcal{A}_t \]

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The refinement update

\[ \phi_0 \quad \phi_t \]

\[ \begin{array}{c}
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\cap
\end{array} \quad \begin{array}{c}
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\end{array} \]

\[ \mathcal{A} \quad \xrightarrow{\quad} \quad \mathcal{A}^t \quad \subseteq \quad \mathcal{A}_t \]

\[ \{X \mid \forall F \in \mathcal{F}_t : 1_F X \in \mathcal{A} \} \quad \text{Tutsch 2006} \]
The refinement update

$\phi_0 \overset{\equiv}{\mapsto} \phi_t$ ‘conditional capital requirement’

Detlefsen & Scandolo 2005; Cheridito Delbean Kupper 2006

$\mathcal{A} \rightarrow \mathcal{A}^t \subseteq \mathcal{A}_t$

$\{X \mid \forall F \in \mathcal{F}_t : 1_F X \in \mathcal{A}\}$ Tutsch 2006
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\[ \phi_0 \quad \Leftrightarrow \quad \phi_t \quad \text{'conditional capital requirement'} \]

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\( \phi_t \) is the only candidate for a weakly time consistent update
The refinement update

\[ \phi_0 \rightarrow \phi_t \]

\[ A \rightarrow A^t \subseteq A_t \]

\{X | \forall F \in \mathcal{F}_t : 1_F X \in A \} \quad \text{Tutsch 2006}

\( \phi_t \) is the only candidate for a weakly time consistent update and can only be consistent if equality holds ("conditional consistency", \( A_t = A^t \))

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The refinement update

\[ \phi_0 \uparrow \phi_t \]

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\( \phi_t \) is the only candidate for a weakly time consistent update and can only be consistent if equality holds ("conditional consistency", \( \mathcal{A}_t = \mathcal{A}^t \))

Extra conditions for sequential consistency in paper

'conditional capital requirement' Detlefsen & Scandolo 2005; Cheridito Delbean Kupper 2006
When can a risk measure be updated consistently?

(i) Determine the refinement update, given by

$$\phi^t_0(X) = \text{ess} \sup \{ Y \in L^\infty_t \mid \phi_0(1_F(X - Y)) \geq 0 \text{ for all } F \in \mathcal{F}_t \}$$

(ii) Check (weak / strong) time consistency
When can a risk measure be updated consistently?

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(ii) Check (weak / strong) time consistency

- Most results in paper rely on (strong) sensitivity
Answer to the main question

When can a risk measure be updated consistently?

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- Compatibility:
  - update to $s$, then update to $t = \text{update to } t$ at once
When can a risk measure be updated consistently?

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(ii) Check (weak / strong) time consistency

- Most results in paper rely on (strong) sensitivity
- Compatibility:
  - update to \( s \), then update to \( t = \) update to \( t \) at once
- Time consistency can be seen as a property of \( \phi_0 \) itself
Coherent case revised

\[ \phi_0 = \inf_{Q \in \mathcal{Q}} E_Q X \]
Coherent case revised

\[ \phi_0 = \inf_{Q \in \mathcal{Q}} E^Q X \quad \phi_t = \inf_{Q \in \mathcal{Q}} E^Q_t X \]
Coherent case revised

\[ \phi_0 = \inf_{Q \in \mathcal{Q}} E^Q X, \quad \phi_t = \inf_{Q \in \mathcal{Q}} E_t^Q X \]

Coincides with the refinement update

Strong: pasting \( \forall Q \in \mathcal{Q} \quad \forall Q' \in \mathcal{Q} : \quad Q' Q_t \in \mathcal{Q} \)
Coherent case revised

\[
\begin{align*}
\phi_0 &= \inf_{Q \in \mathcal{Q}} E^Q X \\
\phi_t &= \inf_{Q \in \mathcal{Q}} E_t^Q X
\end{align*}
\]

Coincides with the refinement update

Strong: pasting \quad \forall Q \in \mathcal{Q} \quad \forall Q' \in \mathcal{Q} : Q' Q_t \in \mathcal{Q}

Sequential: junctioned \quad \forall Q \in \mathcal{Q} \quad \exists Q' \in \mathcal{Q} : Q' Q_t \in \mathcal{Q}
Coherent case revised

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Coincides with the refinement update

**Strong:** pasting \( \forall Q \in \mathcal{Q} \quad \forall Q' \in \mathcal{Q} : \quad Q' Q_t \in \mathcal{Q} \)

**Sequential:** junctioned \( \forall Q \in \mathcal{Q} \quad \exists Q' \in \mathcal{Q} : \quad Q' Q_t \in \mathcal{Q} \)

In *R&S 2005* in a simple setting

Paper on dual characterizations convex risk measures (*L^\infty* setting) in preparation
Example Strong versus Weak

Entropic:

$$- \frac{1}{\beta} \log E[e^{-\beta X}]$$
Example Strong versus Weak

Entropic:

\[-\frac{1}{\beta} \log E[e^{-\beta X}]\]
Example Strong versus Weak

Entropic:

$$-\frac{1}{\beta} \log E[e^{-\beta X}]$$

Consistent updating

\[ \beta_0 \quad \beta_1 \]
Example Strong versus Weak

Entropic:
\[-\frac{1}{\beta} \log \mathbb{E}[e^{-\beta X}]\]

Strong:
(i) choose $\beta_0$ and $\beta_1$
(ii) apply to any position

Consistent updating
Example Strong versus Weak

Entropic:
\[-\frac{1}{\beta} \log E[e^{-\beta X}]\]

Strong:
(i) choose $\beta_0$ and $\beta_1$
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Example Strong versus Weak

Entropic:

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Strong:
(i) choose $\beta_0$ and $\beta_1$
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Example Strong versus Weak

Entropic:

\[ -\frac{1}{\beta} \log E[e^{-\beta X}] \]

Strong:
(i) choose \( \beta_0 \) and \( \beta_1 \)
(ii) apply to any position

Weak:
(i) limit \( \beta_0 + \beta_1 = b \)
(ii) *given* position, apply worst pair
Sequently consistent entropic risk measures

Now $\beta = (\beta_0, \ldots, \beta_{T-1})$, $\beta_t \geq 0$ and $\mathcal{F}_t$-measurable

$$\phi^b(X) = \text{ess inf}\{\phi^b | \Sigma_{t=0}^{T-1} \beta_t = b\}$$

One parameter $b$: overall level of conservatism

$\beta$: pattern of conservatism
Sequentially consistent entropic risk measures

Now \( \beta = (\beta_0, \ldots, \beta_{T-1}) \), \( \beta_t \geq 0 \) and \( \mathcal{F}_t \)-measurable

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\phi^b(X) = \text{ess inf}\{\phi^\beta \mid \sum_{t=0}^{T-1} \beta_t = b\}
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One parameter \( b \): overall level of conservatism
\( \beta \): pattern of conservatism

(a compound risk measure Cheridito & Kupper 2006)
Sequently consistent entropic risk measures

Now $\beta = (\beta_0, \ldots, \beta_{T-1})$, $\beta_t \geq 0$ and $\mathcal{F}_t$-measurable

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\phi^b(X) = \text{ess inf}\{\phi^\beta \mid \sum_{t=0}^{T-1} \beta_t = b\}
$$

One parameter $b$: overall level of conservatism

$\beta$: pattern of conservatism

(a compound risk measure Cheridito & Kupper 2006 )

Easy to compute!

Example: $b = 50$, 100 grid points $b_k$ for beta in (0,50]

Backw. recursively keep track of $\phi^{b_k}_t(X)$ for all grid points $b_k$
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Next use of conservatism after 12% is used

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Consistent updating
(My) Conclusions

1. Risk dynamics $\neq$ Price dynamics
   - far away from strong time consistency
   - strong should be weakened to sequential consistency
   - then still unambiguous updates
   - computations may remain fairly simple: backward recursion in risk profiles $\{\phi_t^c(X)\}_{c \in C}$
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   - WEAK is RIGHT
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2. The extra freedom is crucial
   - only restriction on accumulated conservatism, allowing to combine short and long term considerations in one measure (how exactly??)
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   - allows to first detect the weak spots in a position, then select the most adverse pattern of conservatism
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   - **WEAK is RIGHT**

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     *(how exactly??)*
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   - **WEAK is POWERFUL**
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   - then still unambiguous updates
   - computations may remain fairly simple:
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   - \textbf{WEAK is RIGHT}

2. The extra freedom is crucial
   - only restriction on accumulated conservatism, allowing to combine short and long term considerations in one measure \( \text{(how exactly??)} \)
   - allows to first detect the weak spots in a position, then select the \textbf{most adverse pattern of conservatism}
   - \textbf{WEAK is POWERFUL}