Dynamic Coherent Acceptability Indices

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Performance Measures $f(\text{return, risk})$

- Sharpe Ratio $SR(X) = \frac{\mathbb{E}(X) - r}{\text{Std}(X)}$
- Gain-Loss Ratio $GLR(X) = \frac{\mathbb{E}(X)}{\mathbb{E}(X^-)}$
- Risk Adjusted Return on Capital $RAROC(X) = \frac{\mathbb{E}(X)}{\rho(X)}$
- Treynor Ratios $TR(X) = \frac{\mathbb{E}(X) - r}{\beta(X)}$
- Tilt Coefficient $TC(X) = \sup\{\lambda \in \mathbb{R}_+ \mid \mathbb{E}[X e^{-\lambda X}] \geq 0\}$
- and more

General Desired Properties

- Unitless, Monotone, Quasi-Concave
Objectives

Study these Measures of Performance from abstract and applied probability point of view

- establish a set of axioms (with financial meaning)
- describe all functions that satisfy these axioms (representations theorem)
- cover classical examples
- do it consistently in time (process)
- find new examples / maybe reject some classical ones

\[ f(\text{return, risk, } t) \]
Definition (Cherny and Madan '08)

\( \alpha : \mathcal{X} \rightarrow [0, +\infty] \) is called a coherent \textit{acceptability index (AI)} if the following axioms are satisfied

(A1) \textbf{Monotonicity.} If \( X(\omega) \leq Y(\omega) \) for all \( \omega \in \Omega \), then \( \alpha(X) \leq \alpha(Y) \)

(A2) \textbf{Scale invariance.} For every \( X \in \mathcal{X} \) and \( \lambda > 0 \), \( \alpha(\lambda X) = \alpha(X) \)

(A3) \textbf{Quasi-concavity.} If \( \alpha(X) \geq x \), \( \alpha(Y) \geq x \), then \( \alpha(\lambda X + (1 - \lambda)Y) \geq x \) for all \( \lambda \in [0, 1] \)

(A4) \textbf{Fatou.} If \( |X_n| \leq 1 \), \( \alpha(X_n) \geq x \) and \( X_n \rightarrow X \) in probability, then \( \alpha(X) \geq x \)

- SR - no; GLR - yes; RAROC - yes; TC - no; TVaRAI - yes; etc
- (A1) \( Y \) dominating \( X \) implies \( Y \) is more acceptable than \( X \)
- (A2) cash flows with same structure have same performance
- (A3) diversification does not decrease the performance level
Coherent Acceptability Indices

Static Case

Represenation Theorem (Cherny and Madan '08)

\( \alpha \) is a coherent AI if and only if there exists a family \( (Q_x)_{x \in [0, +\infty]} \) of sets of probability measures, such that \( Q_x \subset Q_y \) for \( x \leq y \) and

\[
\alpha(X) = \sup \left\{ x \in \mathbb{R}_+ : \inf_{Q \in Q_x} \mathbb{E}_Q[X] \geq 0 \right\}
\]

\( \triangleright \) \( GLR(X) = \mathbb{E}[X]/\mathbb{E}[X^-] \) is coherent AI with representation

\[ Q_x = \{ c(1 + Y) : c \in \mathbb{R}_+, 0 \leq Y \leq x, \mathbb{E}[c(1 + Y)] = 1 \} \]

\( \triangleright \) Any coherent acceptability index can be characterized by a family of sets of probability measures.
Performance measurements in dynamic market

- Probability space \((\Omega, \mathcal{F}, \mathbb{P})\)
- Finite time \(\{0, 1, \ldots, T\}\)
- Filtration \(\mathbb{F} = (\mathcal{F}_t)_{t=0}^T\)
- \(D = (D_t)_{t=0}^T\) cash flow
- \(D\) the set of all bounded processes

**Definition**

**Dynamic acceptability index** is a map

\[
\alpha : \{0, \ldots, T\} \times D \times \Omega \to [0, +\infty]
\]
Definition

\( \alpha : \{0, \ldots, T\} \times \mathcal{D} \times \Omega \rightarrow [0, +\infty] \) is called a coherent dynamic acceptability index if it satisfies the following axioms:

(D1) **Adapted:** \( \alpha_t(D, \cdot) \) is \( \mathcal{F}_t \)-measurable

(D2) **Independence of the past:** If there exists \( A \in \mathcal{F}_t \) such that \( 1_A D_s = 1_A D'_s \) for \( s \geq t \), then \( 1_A \alpha_t(D) = 1_A \alpha_t(D') \)

(D3) **Monotonicity:** If \( D_s \geq D'_s \) for some \( D, D' \in \mathcal{D} \) and all \( s \geq t \), then \( \alpha_t(D) \geq \alpha_t(D') \)

(D4) **Scale invariance:** \( \alpha_t(\lambda D, \omega) = \alpha_t(D, \omega) \) for all \( \lambda > 0 \)

(D5) **Quasi-concavity:** If \( \alpha_t(D, \omega) \geq x \), \( \alpha_t(D', \omega) \geq x \), then \( \alpha_t(\lambda D + (1 - \lambda) D', \omega) \geq x \) for all \( \lambda \in [0, 1] \)
Definition Continued

**(D6) Translation Invariance:**

\[ \alpha_t(D + m1_t) = \alpha_t(D + m1_s) \]

for any \( D \in D, \ s \geq t \) and \( m \) \(-\mathcal{F}_t\)-measurable

**(D7) Dynamic consistency:**

Let \( D, D' \in D, \) and \( X \geq 0 \) be \( \mathcal{F}_t \) measurable

(a) If \( D_t \geq 0 \) and \( \alpha_{t+1}(D) \geq X \), then \( \alpha_t(D) \geq X \)

(b) If \( D_t \leq 0 \) and \( \alpha_{t+1}(D) \leq X \), then \( \alpha_t(D) \leq X \)
Theorem (Representation/Duality Theorem)

A function \( \alpha : \{0, 1, \ldots, T\} \times \mathcal{D} \times \Omega \rightarrow [0, +\infty] \) unbounded above is a dynamic coherent acceptability index if and only if there exists a sequence of non-decreasing dynamic coherent risk measures \((\rho^x)_{x \in \mathbb{R}^+}\), such that \( \rho^x_t(D) \geq \rho^y_t(D) \) for \( x \geq y \), and

\[
\alpha_t(D, \omega) = \sup\{x \in \mathbb{R}^+ : \rho^x_t(D, \omega) \leq 0\}.
\]
Theorem (Representation/Duality Theorem)

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\[
\alpha_t(D, \omega) = \sup\{x \in \mathbb{R}_+ : \rho_t^x(D, \omega) \leq 0\}.
\]

The associated risk measures:

\[
\rho_t^x(D) := \inf\{m \in \mathbb{R} : \alpha_t(D + m1_t) \geq x\}, \quad x \in \mathbb{R}^+,
\]

where \( \inf \emptyset = \infty \) and \( \sup \emptyset = 0 \).
**Definition**

Dynamic coherent risk measure is a function \( \rho : \{0, \ldots, T\} \times \mathcal{D} \times \Omega \to \mathbb{R} \) that satisfies the following axioms:

**(A1) Adapted:** For every \( t \in \{0, \ldots, T\} \), and every \( D \in \mathcal{D} \), \( \rho_t(D) \) is \( \mathcal{F}_t \)-measurable

**(A2) Independence of the past:** If there exists \( A \in \mathcal{F}_t \) such that 
\[ 1_A D_s = 1_A D'_s \] for all \( s \geq t \), then \( 1_A \rho_t(D) = 1_A \rho_t(D') \)

**(A3) Monotonicity:** If \( D_s \geq D'_s \) for some \( D, D' \in \mathcal{D} \), and for all \( s \geq t \), then \( \rho_t(D) \leq \rho_t(D') \)

**(A4) Homogeneity:** \( \rho_t(\lambda D) = \lambda \rho_t(D) \) for every \( \lambda \geq 0 \), \( D \in \mathcal{D} \), \( t \in \{0, \ldots, T\} \)

**(A5) Subadditivity:** \( \rho_t(D + D') \leq \rho_t(D) + \rho_t(D') \) for every \( D, D' \in \mathcal{D} \), \( t \in \{0, \ldots, T\} \)

**(A6) Translation Invariance:** \( \rho_t(D + m1_s) = \rho_t(D) - m \) for every \( D \in \mathcal{D} \), an \( \mathcal{F}_t \)-measurable random variable \( m \), and for all \( s \geq t \)
Dynamic Coherent Risk Measures

**Definition**

(A7) **Dynamic consistency:** For every $D \in \mathcal{D}$, we have

$$
\min_{\omega} \rho_{t+1}(D, \omega) - D_t \leq \rho_t(D) \leq \max_{\omega} \rho_{t+1}(D, \omega) - D_t
$$

(A7') **Riedel 2004:**

If $D_t = D'_t$ and $\rho_{t+1}(D, \omega) = \rho_{t+1}(D', \omega)$ for all $\omega \in \Omega$, then $\rho_t(D, \omega) = \rho_t(D', \omega)$ for all $\omega \in \Omega$

(A7'') **superposition:** $\rho_t(D) = \rho_t(D_t 1_t - \rho_{t+1}(D) 1_{t+1})$

▷ (A1)-(A6) and (A7) $\Rightarrow$ (A7)

▷ (A1)-(A6) and (A7'') $\Rightarrow$ (A7)
Dynamic Set of Probability Measures

Definition

\( Q \subset \mathcal{P} \) is called dynamic set of probability measures if

\[
\inf_{Q \in \mathcal{Q}} \mathbb{E}_{Q} \left[ D \mid \mathcal{F}_{t} \right] = \inf_{Q \in \mathcal{Q}} \mathbb{E}_{Q} \left[ \inf_{M \in \mathcal{Q}} \mathbb{E}_{M} \left[ D \mid \mathcal{F}_{t+1} \right] \mid \mathcal{F}_{t} \right] \quad \forall t, D
\]

- Two trivial examples: singleton set \( Q = \{Q\} \) and \( \mathcal{P} \)
- Two non-trivial examples: for \( x \geq 0 \), define

\[
Q_{x}^{u} := \{ Q \in \mathcal{P} \mid \mathbb{E}_{P} \left[ \frac{dQ}{dP} \right] \mathcal{F}_{j} \leq (1 + x) \mathbb{E}_{P} \left[ \frac{dQ}{dP} \right] \mathcal{F}_{j-1} \} \quad \forall j = 1, \ldots, T
\]

\[
Q_{x}^{l} := \{ Q \in \mathcal{P} \mid \mathbb{E}_{Q} \left[ \frac{dP}{dQ} \right] \mathcal{F}_{j} \leq (1 + x) \mathbb{E}_{Q} \left[ \frac{dP}{dQ} \right] \mathcal{F}_{j-1} \} \quad \forall j = 1, \ldots, T
\]
Theorem

Assume that \( \{ Q_x \}_{x \in \mathbb{R}_+} \) is a family of dynamic sets of probability measures, such that \( Q_x \subset Q_y \) for \( x \leq y \) and define

\[
\alpha_t(D) := \sup \left\{ x \in \mathbb{R}_+ : \inf_{Q \in Q_x} \mathbb{E}_Q \left[ \sum_{s=t}^{T} D_t \mid \mathcal{F}_t \right] \geq 0 \right\}
\]

Then \( \alpha \) is a dynamic coherent acceptability index.

- Static case is a particular case
- \( \rho^x_t(D) := - \inf_{Q \in Q_x} \mathbb{E}_Q \left[ \sum_{i=t}^{T} D_i \mid \mathcal{F}_t \right] \) is a dynamic coherent risk measure
- The existence of family \( Q_x \) is guaranteed by \( \{ Q^u_x \} \) and \( \{ Q^l_x \} \)
Dynamic Gain-Loss Ratio

\[ dGLR_t(D) = \begin{cases} 
\frac{\mathbb{E}[\sum_{s=t}^{T} D_s | \mathcal{F}_t]}{\mathbb{E}((\sum_{s=t}^{T} D_s) - |\mathcal{F}_t|)} & \text{if } \mathbb{E}[\sum_{s=t}^{T} D_s | \mathcal{F}_t] > 0 \\
0 & \text{otherwise}
\end{cases} \]

Dynamic RAROC

Given a dynamic set of probability measures \( \mathcal{Q} \) with \( \mathbb{P} \in \mathcal{Q} \), define

\[ dRAROC_t(D) = 1\{\mathbb{E}[\sum_{s=t}^{T} D_s | \mathcal{F}_t] > 0\} \frac{\mathbb{E}[\sum_{s=t}^{T} D_s | \mathcal{F}_t]}{- \inf_{Q \in \mathcal{Q}} \mathbb{E}_Q[\sum_{s=t}^{T} D_s | \mathcal{F}_t]} \]

(convention \( dRAROC_t(D) = +\infty \) if \( \inf_{Q \in \mathcal{Q}} \mathbb{E}_Q[\sum_{s=t}^{T} D_s | \mathcal{F}_t] \geq 0 \))

Dynamic Sharpe Ratio

\[ dSR_t(D) = \begin{cases} 
\frac{\mathbb{E}[\sum_{s=t}^{T} D_s | \mathcal{F}_t]}{\text{Std}[\sum_{s=t}^{T} D_s | \mathcal{F}_t]} & \text{if } \mathbb{E}[\sum_{s=t}^{T} D_s | \mathcal{F}_t] > 0 \\
0 & \text{otherwise}
\end{cases} \]
Example 1: dGLR vs dSR
Example 2: dRAROC is not dynamic consistent

(a) If $D_t \geq 0$ and $\alpha_{t+1}(D) \geq X$, then $\alpha_t(D) \geq X$
(b) If $D_t \leq 0$ and $\alpha_{t+1}(D) \leq X$, then $\alpha_t(D) \leq X$