Analysis of Fourier Transform Valuation Formulas and Applications

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Volatility surfaces of foreign exchange and interest rate options

- Volatilities vary in strike (smile)
- Volatilities vary in time to maturity (term structure)
- Volatility clustering
Fourier and Laplace based valuation formulas

Carr and Madan (1999)
Raible (2000)
Borovkov and Novikov (2002): exotic options
Lee (2004): discretization error in fast Fourier transform
Hubalek and Kallsen (2005): options on several assets
Biagini, Bregman, and Meyer-Brandis (2008): indices
Hurd and Zhou (2009): spread options
Eberlein and Kluge (2006): interest rate derivatives
Eberlein, Kluge, and Schönbucher (2006): credit default swaptions
Harmonic analysis (Parseval’s formula)
The model

Exponential semimartingale model

\[ B_T = (\Omega, \mathcal{F}, \mathcal{F}, P) \] stochastic basis, where \( \mathcal{F} = \mathcal{F}_T \) and \( \mathcal{F} = (\mathcal{F}_t)_{0 \leq t \leq T} \).

Price process of a financial asset as exponential semimartingale

\[ S_t = S_0 e^{H_t}, \quad 0 \leq t \leq T. \tag{1} \]

\( H = (H_t)_{0 \leq t \leq T} \) semimartingale with canonical representation

\[ H = B + H^c + h(x) \ast (\mu^H - \nu) + (x - h(x)) \ast \mu^H. \tag{2} \]

For the processes \( B, C = \langle H^c \rangle \), and the measure \( \nu \) we use the notation

\[ \mathbb{T}(H|P) = (B, C, \nu) \]

which is called the \textit{triplet of predictable characteristics} of \( H \).
Alternative model description

\[ \mathcal{E}(X) = (\mathcal{E}(X)_t)_{0 \leq t \leq T} \quad \text{stochastic exponential} \]

\[ S_t = \mathcal{E}(\tilde{H})_t, \quad 0 \leq t \leq T \]
\[ dS_t = S_t d\tilde{H}_t \]

where

\[ \tilde{H}_t = H_t + \frac{1}{2} \langle H^c \rangle_t + \int_0^t \int_{\mathbb{R}} (e^x - 1 - x) \mu(H)(ds, dx) \]

Note

\[ \mathcal{E}(\tilde{H})_t = \exp \left( \tilde{H}_t - \frac{1}{2} \langle H^c \rangle_t \right) \prod_{0 < s \leq t} (1 + \Delta \tilde{H}_s) \exp(-\Delta \tilde{H}_s) \]

Asset price positive only if \( \Delta \tilde{H} > -1 \).
Martingale modeling

Let $\mathcal{M}_{\text{loc}}(P)$ be the class of local martingales.

**Assumption (ES)**

The process $1_{\{x>1\}} e^x \ast \nu$ has bounded variation.

Then

$$ S = S_0 e^H \in \mathcal{M}_{\text{loc}}(P) \iff B + \frac{C}{2} + (e^x - 1 - h(x)) \ast \nu = 0. \quad (3) $$

Throughout, we assume that $P$ is an equivalent martingale measure for $S$.

By the *Fundamental Theorem of Asset Pricing*, the value of an option on $S$ equals the *discounted expected payoff* under this martingale measure.

We assume zero interest rates.
Supremum and infimum processes

Let $X = (X_t)_{0 \leq t \leq T}$ be a stochastic process. Denote by

$$
\overline{X}_t = \sup_{0 \leq u \leq t} X_u \quad \text{and} \quad \underline{X}_t = \inf_{0 \leq u \leq t} X_u
$$

the supremum and infimum process of $X$ respectively. Since the exponential function is monotone and increasing

$$
\overline{S}_T = \sup_{0 \leq t \leq T} S_t = \sup_{0 \leq t \leq T} \left( S_0 e^{H_t} \right) = S_0 e^{\sup_{0 \leq t \leq T} H_t} = S_0 e^{\overline{H}_T}. \quad (4)
$$

Similarly

$$
\underline{S}_T = S_0 e^{\underline{H}_T}. \quad (5)
$$
Valuation formulas – payoff functional

We want to price an option with payoff $\Phi(S_t, 0 \leq t \leq T)$, where $\Phi$ is a measurable, non-negative functional.

Separation of payoff function from the underlying process:

**Example**

*Fixed strike lookback option*

\[
(S_T - K)^+ = (S_0 e^{H_T} - K)^+ = (e^{H_T + \log S_0} - K)^+
\]

1. The *payoff function* is an arbitrary function $f: \mathbb{R} \to \mathbb{R}_+$; for example $f(x) = (e^x - K)^+$ or $f(x) = 1_{\{e^x > B\}}$, for $K, B \in \mathbb{R}_+$.

2. The *underlying process* denoted by $X$, can be the log-asset price process or the supremum/infimum or an average of the log-asset price process (e.g. $X = H$ or $X = \overline{H}$).
Valuation formulas

Consider the option price as a function of $S_0$ or better of $s = -\log S_0$

$X$ driving process ($X = H, \bar{H}, \underline{H}$, etc.)

$\Rightarrow \Phi(S_0 e^{H_t}, 0 \leq t \leq T) = f(X_T - s)$

Time-0 price of the option (assuming $r \equiv 0$)

$\nabla_f(X; s) = E[\Phi(S_t, 0 \leq t \leq T)] = E[f(X_T - s)]$

Valuation formulas based on Fourier and Laplace transforms

Carr and Madan (1999) plain vanilla options

Raible (2000) general payoffs, Lebesgue densities

In these approaches: Some sort of continuity assumption (payoff or random variable)
Valuation formulas – assumptions

\( M_{X_T} \) moment generating function of \( X_T \)
\[ g(x) = e^{-Rx}f(x) \] (for some \( R \in \mathbb{R} \)) dampered payoff function
\( L_{bc}^1(\mathbb{R}) \) bounded, continuous functions in \( L^1(\mathbb{R}) \)

**Assumptions**

(C1) \( g \in L_{bc}^1(\mathbb{R}) \)
(C2) \( M_{X_T}(R) \) exists
(C3) \( \hat{g} \in L^1(\mathbb{R}) \)
Valuation formulas

**Theorem**

Assume that (C1)–(C3) are in force. Then, the price $V_f(X; s)$ of an option on $S = (S_t)_{0 \leq t \leq T}$ with payoff $f(X_T)$ is given by

$$V_f(X; s) = e^{-Rs} \frac{2\pi}{2\pi} \int_{\mathbb{R}} e^{ius} \varphi_{X_T}(-u - iR) \hat{f}(u + iR) du,$$

(6)

where $\varphi_{X_T}$ denotes the extended characteristic function of $X_T$ and $\hat{f}$ denotes the Fourier transform of $f$. 
Discussion of assumptions

Alternative choice: \((C1')\) \(g \in L^1(\mathbb{R})\)

\((C3')\) \(e^{R \cdot P_T} \in L^1(\mathbb{R})\)

\((C3')\) \(\Rightarrow e^{R \cdot P_T}\) has a cont. bounded Lebesgue density

Recall: \((C3)\) \(\hat{g} \in L^1(\mathbb{R})\)

Sobolov space

\[ H^1(\mathbb{R}) = \{ g \in L^2(\mathbb{R}) \mid \partial g \text{ exists and } \partial g \in L^2(\mathbb{R}) \} \]

**Lemma**

\(g \in H^1(\mathbb{R}) \Rightarrow \hat{g} \in L^1(\mathbb{R})\)

Similar for the Sobolev–Slobodeckij space \(H^S(\mathbb{R}) \ (s > \frac{1}{2})\)
Examples of payoff functions

**Example (Call and put option)**

*Call payoff* \( f(x) = (e^x - K)^+ \), \( K \in \mathbb{R}_+ \),

\[
\hat{f}(u + iR) = \frac{K^{1+iu-R}}{(iu - R)(1 + iu - R)}, \quad R \in I_1 = (1, \infty). \tag{7}
\]

Similarly, if \( f(x) = (K - e^x)^+ \), \( K \in \mathbb{R}_+ \),

\[
\hat{f}(u + iR) = \frac{K^{1+iu-R}}{(iu - R)(1 + iu - R)}, \quad R \in I_1 = (-\infty, 0). \tag{8}
\]
Example (Digital option)

**Call payoff** \( \mathbb{1}_{\{e^x > B\}}, B \in \mathbb{R}_+ \).

\[
\hat{f}(u + iR) = -B^{iu-R} \frac{1}{iu-R}, \quad R \in I_1 = (0, \infty).
\] (9)

Similarly, for the payoff \( f(x) = \mathbb{1}_{\{e^x < B\}}, B \in \mathbb{R}_+ \),

\[
\hat{f}(u + iR) = B^{iu-R} \frac{1}{iu-R}, \quad R \in I_1 = (-\infty, 0).
\] (10)

Example (Double digital option)

The payoff of a double digital option is \( \mathbb{1}_{\{B < e^x < \bar{B}\}}, B, \bar{B} \in \mathbb{R}_+ \).

\[
\hat{f}(u + iR) = \frac{1}{iu-R} \left( \bar{B}^{iu-R} - B^{iu-R} \right), \quad R \in I_1 = \mathbb{R} \setminus \{0\}.
\] (11)
Example (Asset-or-nothing digital)

Payoff \( f(x) = e^x \mathbb{1}_{\{e^x > B\}} \)

\[ \hat{f}(u + iR) = -\frac{B^{1+iu-R}}{1 + iu - R}, \quad R \in I_1 = (1, \infty) \]

Similarly \( f(x) = e^x \mathbb{1}_{\{e^x < B\}} \)

\[ \hat{f}(u + iR) = \frac{B^{1+iu-R}}{1 + iu - R}, \quad R \in I_1 = (-\infty, 1) \]

Example (Self-quanto option)

Call payoff \( f(x) = e^x (e^x - K)^+ \)

\[ \hat{f}(u + iR) = \frac{K^{2+iu-R}}{(1 + iu - R)(2 + iu - R)}, \quad R \in I_1 = (2, \infty) \]
Non-path-dependent options

European option on an asset with price process \( S_t = e^{H_t} \)

Examples: call, put, digitals, asset-or-nothing, double digitals, self-quanto options

\[ X_T \equiv H_T, \quad \text{i.e. we need } \varphi_{H_T} \]


\[
\varphi_{H_1}(u) = e^{iu\mu} \left( \frac{\alpha^2 - \beta^2}{\alpha^2 - (\beta + iu)^2} \right)^{\lambda/2} \frac{K_{\lambda}(\delta \sqrt{\alpha^2 - (\beta + iu)^2})}{K_{\lambda}(\delta \sqrt{\alpha^2 - \beta^2})}
\]

\[ l_2 = (-\alpha - \beta, \alpha - \beta) \]

\[ \varphi_{H_T}(u) = (\varphi_{H_1}(u))^T \]

similar: NIG, CGMY, Meixner
Non-path-dependent options II

Stochastic volatility Lévy models: Carr, Geman, Madan, Yor (2003)  
Eberlein, Kallsen, Kristen (2003)

Stochastic clock  \[ Y_t = \int_0^t y_s ds \quad (y_s > 0) \]
e.g. CIR process

\[ dy_t = K(\eta - y_t)dt + \lambda y_t^{1/2} dW_t \]

Define for a pure jump Lévy process \( X = (X_t)_{t \geq 0} \)

\[ H_t = X_{Y_t} \quad (0 \leq t \leq T) \]

Then

\[ \varphi_{H_t}(u) = \frac{\varphi_{Y_t}(-i\varphi_{X_t}(u))}{(\varphi_{Y_t}(-iu \varphi_{X_t}(-i)))^{iu}} \]
## Classification of option types

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<th>( S_t = S_0 e^{H_t} )</th>
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<td>( f(x) = (e^x - K)^+ )</td>
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<td></td>
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</tbody>
</table>
Valuation formula for the last case

Payoff function $f$ maybe discontinuous

$P_{X_T}$ does not necessarily possess a Lebesgue density

Assumption

(D1) $g \in L^1(\mathbb{R})$

(D2) $M_{X_T}(R)$ exists

Theorem

Assume (D1)–(D2) then

$$V_f(X; s) = \lim_{A \to \infty} \frac{e^{-Rs}}{2\pi} \int_{-A}^{A} e^{-ius} \varphi_{X_T}(u - iR) \hat{f}(iR - u) \, du$$

if $V_f(X; \cdot)$ is of bounded variation in a neighborhood of $s$ and $V_f(X; \cdot)$ is continuous at $s$. 
Options on multiple assets

Basket options

Options on the minimum: \((S_T^1 \wedge \cdots \wedge S_T^d - K)^+\)

Multiple functionals of one asset

Barrier options: \((S_T - K)^+ \mathbb{1}_{\{S_T > B\}}\)

Slide-in or corridor options: \((S_T - K)^+ \sum_{i=1}^{N} \mathbb{1}_{\{L < S_{T_i} < H\}}\)

Modelling:
\[
S_t^i = S_0^i \exp(H_t^i) \quad (1 \leq i \leq d)
\]
\[
X_T = \Psi(H_t \mid 0 \leq t \leq T)
\]
\[
f : \mathbb{R}^d \rightarrow \mathbb{R}_+
\]
\[
g(x) = e^{-\langle R, x \rangle} f(x) \quad (x \in \mathbb{R}^d)
\]

Assumptions:

(A1) \(g \in L^1(\mathbb{R}^d)\)

(A2) \(M_{X_T}(R)\) exists

(A3) \(\hat{\varrho} \in L^1(\mathbb{R}^d)\) where \(\varrho(dx) = e^{\langle R, x \rangle} P_X(dx)\)
Theorem

*If the asset price processes are modeled as exponential semimartingale processes such that $S_i \in \mathcal{M}_{1\text{oc}}(P)$ (1 ≤ i ≤ d) and conditions (A1)–(A3) are in force, then*

$$V_f(X; s) = \frac{e^{-\langle R, s \rangle}}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\langle u, s \rangle} \mathcal{M}_{X_T}(R + iu) f(iR - u) du$$

Remark

*When the payoff function is discontinuous and the driving process does not possess a Lebesgue density $\rightarrow L^2$-limit result*
Sensitivities – Greeks

\[ V_f(X; S_0) = \frac{1}{2\pi} \int_{\mathbb{R}} S_0^{R-\text{i}u} M_{X_T}(R - \text{i}u) \hat{f}(u + \text{i}R) du \]

Delta of an option

\[ \Delta_f(X; S_0) = \frac{\partial V(X; S_0)}{\partial S_0} = \frac{1}{2\pi} \int_{\mathbb{R}} S_0^{R-1-\text{i}u} M_{X_T}(R - \text{i}u) \frac{\hat{f}(u + \text{i}R)}{(R - \text{i}u)^{-1}} du \]

Gamma of an option

\[ \Gamma_f(X; S_0) = \frac{\partial^2 V_f(X; S_0)}{\partial^2 S_0} = \frac{1}{2\pi} \int_{\mathbb{R}} S_0^{R-2-\text{i}u} M_{X_T}(R - \text{i}u) \frac{\hat{f}(u + \text{i}R) (R - 1 - \text{i}u)^{-1} (R - \text{i}u)^{-1}}{du} \]
Numerical examples

Option prices in the 2d Black-Scholes model with negative correlation.

Option prices in the 2d stochastic volatility model.

Option prices in the 2d GH model with positive (left) and negative (right) correlation.
Lévy processes

Let $L = (L_t)_{0 \leq t \leq T}$ be a Lévy process with triplet of local characteristics $(b, c, \lambda)$, i.e. $B_t(\omega) = bt$, $C_t(\omega) = ct$, $\nu(\omega; dt, dx) = dt\lambda(dx)$, $\lambda$ Lévy measure.

**Assumption $(EM)$**

There exists a constant $M > 1$ such that

$$\int_{\{|x|>1\}} e^{ux} \lambda(dx) < \infty, \quad \forall u \in [-M, M].$$

Using $(EM)$ and Theorems 25.3 and 25.17 in Sato (1999), we get that

$$E[e^{uL_t}] < \infty, \quad E[e^{u\bar{L}_t}] < \infty \quad \text{and} \quad E[e^{uL_t}] < \infty$$

for all $u \in [-M, M]$. 
On the characteristic function of the supremum I

Proposition

Let $L = (L_t)_{0 \leq t \leq T}$ be a Lévy process that satisfies assumption $(EM)$. Then, the characteristic function $\varphi_{\tilde{L}_t}$ of $\tilde{L}_t$ has an analytic extension to the half plane $\{z \in \mathbb{C} : -M < \Im z < \infty\}$ and can be represented as a Fourier integral in the complex domain

$$\varphi_{\tilde{L}_t}(z) = E[e^{iz\tilde{L}_t}] = \int_{\mathbb{R}} e^{izx} P_{\tilde{L}_t}(dx).$$
Fluctuation theory for Lévy processes

Theorem
(Extension of Wiener–Hopf to the complex plane)

Let \( L \) be a Lévy process. The Laplace transform of \( \bar{L} \) at an independent and exponentially distributed time \( \theta, \theta \sim \text{Exp}(q) \), can be identified from the Wiener–Hopf factorization of \( L \) via

\[
E[e^{-\beta \bar{L}\theta}] = \int_0^\infty qE[e^{-\beta \bar{L}t}]e^{-qt} \, dt = \frac{\kappa(q, 0)}{\kappa(q, \beta)}
\]  

for \( q > \alpha^*(M) \) and \( \beta \in \{ \beta \in \mathbb{C} | \Re(\beta) > -M \} \) where \( \kappa(q, \beta), \) is given by

\[
\kappa(q, \beta) = k \exp \left( \int_0^\infty \int_0^\infty (e^{-t} - e^{-qt-\beta x}) \frac{1}{t} P_L(dx) \, dt \right).
\]
On the characteristic function of the supremum II

**Theorem**

Let $L = (L_t)_{0 \leq t \leq T}$ be a Lévy process satisfying assumption $(EM)$. The Laplace transform of $L_t$ at a fixed time $t$, $t \in [0, T]$, is given by

$$E[e^{-\beta L_t}] = \lim_{A \to \infty} \frac{1}{2\pi} \int_{-A}^{A} \frac{e^{t(Y+iv)}}{Y + iv} \frac{\kappa(Y + iv, 0)}{\kappa(Y + iv, \beta)} dv,$$

for $Y > \alpha^*(M)$ and $\beta \in \mathbb{C}$ with $\Re \beta \in (-M, \infty)$.

**Remark**

Note that $\beta = -iz$ provides the characteristic function.
Application to lookback options

Fixed strike lookback call: $(\bar{S}_T - K)^+$ (analogous for lookback put).

Combining the results, we get

$$C_T(\bar{S}; K) = \frac{1}{2\pi} \int_{\mathbb{R}} S_0^{R-iu} \varphi^{-}_{LT}(-u - iR) \frac{K^{1+iu-R}}{(iu - R)(1 + iu - R)} du \quad (16)$$

where

$$\varphi^{-}_{LT}(-u - iR) = \lim_{A \to \infty} \frac{1}{2\pi} \int_{-A}^{A} e^{T(Y + iv)} \frac{\kappa(Y + iv, 0)}{\kappa(Y + iv, iu - R)} dv \quad (17)$$

for $R \in (1, M)$ and $Y > \alpha^*(M)$.

- The floating strike lookback option, $(\bar{S}_T - S_T)^+$, is treated by a duality formula (Eb., Papapantoleon (2005)).
One-touch options

One-touch call option: \( 1_{\{S_T > B\}} \).

Driving Lévy process \( L \) is assumed to have infinite variation or has infinite activity and is regular upwards. \( L \) satisfies assumption (EM), then

\[
\begin{align*}
\text{DC}_T(S; B) &= \lim_{A \to \infty} \frac{1}{2\pi} \int_{-A}^{A} S_0^{R+iu} \varphi_L(u - iR) \frac{B^{-R-iu}}{R + iu} \, du \\
&= P(L_T > \log(B/S_0)) \\
&\quad \text{for } R \in (0, M).
\end{align*}
\]
Equity default swap (EDS)

- Fixed premium exchanged for payment at “default”
- default: drop of stock price by 30% or 50% of $S_0 \to$ first passage time
- fixed leg pays premium $K$ at times $T_1, \ldots, T_N$, if $T_i \leq \tau_B$
- if $\tau_B \leq T$: protection payment $C$, paid at time $\tau_B$
- premium of the EDS chosen such that initial value equals 0; hence

$$K = \frac{CE \left[ e^{-r\tau_B} \mathbb{1}_{\{\tau_B \leq T\}} \right]}{\sum_{i=1}^N E \left[ e^{-rT_i} \mathbb{1}_{\{\tau_B > T_i\}} \right]}.$$

(19)

- Calculations similar to touch options, since $\mathbb{1}_{\{\tau_B \leq T\}} = \mathbb{1}_{\{S_T \leq B\}}$. 

The model
Valuation
Payoff functions and processes
Valuation continued
Exotic options
Interest rate derivatives
References
Basic interest rates

\[ B(t, T) : \text{ price at time } t \in [0, T] \text{ of a default-free zero coupon bond with maturity } T \in [0, T^*] \quad (B(T, T) = 1) \]

\[ f(t, T) : \text{ instantaneous forward rate} \]

\[ B(t, T) = \exp \left( - \int_t^T f(t, u) \, du \right) \]

\[ L(t, T) : \text{ default-free forward Libor rate for the interval } T \text{ to } T + \delta \text{ as of time } t \leq T \quad (\delta\text{-forward Libor rate}) \]

\[ L(t, T) := \frac{1}{\delta} \left( \frac{B(t, T)}{B(t, T + \delta)} - 1 \right) \]

\[ F_B(t, T, U) : \text{ forward price process for the two maturities } T < U \]

\[ F_B(t, T, U) := \frac{B(t, T)}{B(t, U)} \]

\[ 1 + \delta L(t, T) = \frac{B(t, T)}{B(t, T + \delta)} = F_B(t, T, T + \delta) \]
Dynamics of the forward rates

(Eb–Raible (1999), Eb–Özkan (2003),

\[ df(t, T) = \alpha(t, T) \, dt - \sigma(t, T) \, dL_t \quad (0 \leq t \leq T \leq T^*) \]

\( \alpha(t, T) \) and \( \sigma(t, T) \) satisfy measurability and boundedness conditions
and \( \alpha(s, T) = \sigma(s, T) = 0 \) for \( s > T \)

Define \( A(s, T) = \int_{s \wedge T}^{T} \alpha(s, u) \, du \) and \( \Sigma(s, T) = \int_{s \wedge T}^{T} \sigma(s, u) \, du \)

Assume \( 0 \leq \Sigma^i(s, T) \leq M \quad (1 \leq i \leq d) \)

For most purposes we can consider deterministic \( \alpha \) and \( \sigma \)
Implications

Savings account and default-free zero coupon bond prices are given by

\[ B_t = \frac{1}{B(0, t)} \exp \left( \int_0^t A(s, T) \, ds - \int_0^t \Sigma(s, t) \, dL_s \right) \quad \text{and} \]

\[ B(t, T) = B(0, T)B_t \exp \left( -\int_0^t A(s, T) \, ds + \int_0^t \Sigma(s, T) \, dL_s \right). \]

If we choose \( A(s, T) = \theta_s(\Sigma(s, T)) \), then bond prices, discounted by the savings account, are martingales.

In case \( d = 1 \), the martingale measure is unique (see Eberlein, Jacod, and Raible (2004)).
Key tool

$L = (L^1, \ldots, L^d)$ \quad \text{\textit{d}-dimensional time-inhomogeneous \text{Lévy process}}

\[ E[\exp(i\langle u, L_t \rangle)] = \exp \int_0^t \theta_s(iu) \, ds \quad \text{where} \]

\[ \theta_s(z) = \langle z, b_s \rangle + \frac{1}{2} \langle z, c_s z \rangle + \int_{\mathbb{R}^d} \left( e^{\langle z, x \rangle} - 1 - \langle z, x \rangle \right) F_s(dx) \]

in case $L$ is a (time-homogeneous) \text{Lévy process}, $\theta_s = \theta$ is the cumulant (log-moment generating function) of $L_1$.

\begin{table}[h]
\begin{tabular}{|l|}
\hline
\textbf{Proposition} & Eberlein, Raible (1999) \\
\hline
\textbf{Suppose $f : \mathbb{R}_+ \to \mathbb{C}^d$ is a continuous function such that $|R(f^i(x))| \leq M$ for all $i \in \{1, \ldots, d\}$ and $x \in \mathbb{R}_+$, then} & \\
\hline
$E \left[ \exp \left( \int_0^t f(s) \, dL_s \right) \right] = \exp \left( \int_0^t \theta_s(f(s)) \, ds \right)$ & \\
\hline
\end{tabular}
\end{table}

Take $f(s) = \sum(s, T)$ for some $T \in [0, T^*]$. 

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Pricing of European options

\[ B(t, T) = B(0, T) \exp \left[ \int_0^t (r(s) + \theta_s(\Sigma(s, T))) \, ds + \int_0^t \Sigma(s, T) \, dL_s \right] \]

where \( r(t) = f(t, t) \) short rate

\[ V(0, t, T, w) \text{ time-0-price of a European option with maturity } t \text{ and payoff } w(B(t, T), K) \]

\[ V(0, t, T, w) = \mathbb{E}_{\mathbb{P}^*} \left[ B_t^{-1} w(B(t, T), K) \right] \]

Volatility structures

\[ \Sigma(t, T) = \frac{\hat{\sigma}}{a} (1 - \exp(-a(T - t))) \quad (\text{Vasiček}) \]

\[ \Sigma(t, T) = \tilde{\sigma}(T - t) \quad (\text{Ho–Lee}) \]

Fast algorithms for Caps, Floors, Swaptions, Digitals, Range options
Pricing formula for caps
(Eberlein, Kluge (2006))

\[ w(B(t, T), K) = (B(t, T) - K)^+ \]

Call with strike \( K \) and maturity \( t \) on a bond that matures at \( T \)

\[ C(0, t, T, K) = \mathbb{E}_{\mathbb{P}^*} [B_t^{-1}(B(t, T) - K)^+] \]
\[ = B(0, t)\mathbb{E}_{\mathbb{P}_t} [(B(t, T) - K)^+] \]

Assume \( X = \int_0^t (\Sigma(s, T) - \Sigma(s, t))dL_s \) has a Lebesgue density, then

\[ C(0, t, T, K) = \frac{1}{2\pi} KB(0, t) \exp(R\xi) \]
\[ \times \int_{-\infty}^{\infty} e^{iu\xi}(R + iu)^{-1}(R + 1 + iu)^{-1} M_t^X(-R - iu)du \]

where \( \xi \) is a constant and \( R < -1 \).

Analogous for the corresponding put and for swaptions.
References

References (cont.)


