On the Convergence of Higher Order Hedging Schemes

Magnus Wiktorsson  Mats Brodén

Centre for Mathematical Sciences
Mathematical Statistics
Lund University
magnusw@maths.lth.se

June 23, 2010
Outline

1. Introduction
2. Setting
3. Numerical Experiment
4. Results
5. Conclusions
Hedging Error

Arbitrage Theory in Continuous Time: In a complete market setting every contingent claim can be replicated by continuously trade in the underlying.

In practice: Continuous trading is impossible.

Hedging error $R$, i.e. the value of the hedge portfolio differ by some amount $R$ from the value of the derivative.
Hedging Error

**Arbitrage Theory in Continuous Time**: In a complete market setting every contingent claim can be replicated by continuously trade in the underlying.

**In practice**: Continuous trading is impossible.

⇓

Hedging error $\mathcal{R}$, i.e. the value of the hedge portfolio differ by some amount $\mathcal{R}$ from the value of the derivative.
Hedging Error

Arbitrage Theory in Continuous Time: In a complete market setting every contingent claim can be replicated by continuously trade in the underlying.

In practice: Continuous trading is impossible.

Hedging error $R$, i.e. the value of the hedge portfolio differ by some amount $R$ from the value of the derivative.
Setting

Risky asset under $\mathbb{Q}$:
$$dX(t) = rX(t)dt + \sigma(X(t))X(t)dW(t).$$

Bank account:
$$dB(t) = rB(t)dt.$$

Derivative prices:
$$F_i(t, X(t)) = e^{-r(T_i-t)}\mathbb{E}[\Phi_i(X(T_i))|\mathcal{F}_t], \ i \in \{1, 2\}.$$

Assumptions:

Let $\tilde{\sigma}(y) = \sigma(e^y)$.

A1.  
(i) There is a positive constant $\sigma_0$ such that $\tilde{\sigma}(y) \geq \sigma_0$ for all $y \in \mathbb{R}$.

(ii) The function $\tilde{\sigma}$ is bounded, uniformly Lipschitz continuous in compact subsets of $\mathbb{R}$ and uniformly Hölder continuous.

A2. The functions $(\partial^k/\partial y^k)\tilde{\sigma}(y), \ i \in \{1, 2\}$, are bounded.

A3. $\Phi_1(x) = (x - K_1)^+, \ \Phi_2(x) = (x - K_2)^+$ and $T_2 > T_1$. 
Setting

Risky asset under $\mathbb{Q}$: $dX(t) = rX(t)dt + \sigma(X(t))X(t)dW(t)$.

Bank account: $dB(t) = rB(t)dt$.

Derivative prices: $F_i(t, X(t)) = e^{-r(T_i - t)}\mathbb{E}[\Phi_i(X(T_i))|\mathcal{F}_t]$, $i \in \{1, 2\}$.

Assumptions:

Let $\tilde{\sigma}(y) = \sigma(e^y)$.

A1. (i) There is a positive constant $\sigma_0$ such that $\tilde{\sigma}(y) \geq \sigma_0$ for all $y \in \mathbb{R}$.
   (ii) The function $\tilde{\sigma}$ is bounded, uniformly Lipschitz continuous in compact subsets of $\mathbb{R}$ and uniformly Hölder continuous.

A2. The functions $\left(\frac{\partial^k}{\partial y^k}\right)\tilde{\sigma}(y)$, $i \in \{1, 2, 3, 4\}$, are bounded.

A3. $\Phi_1(x) = (x - K_1)^+$, $\Phi_2(x) = (x - K_2)^+$ and $T_2 > T_1$. 
Δ-hedging

Find a self-financing portfolio \( \{ h^X, h^B \} \) such that

\[
h^X(t)X(t) + h^B B(t) = F_1(t, X(t))
\]

for all \( t \in [0, T_1] \).

Solution: let \( h^X(t) = \frac{\partial F_1}{\partial x}(t, X(t)) = F_{1,x}(t, X(t)) \).
**Δ-hedging**

Find a self-financing portfolio \( \{ h^X, h^B \} \) such that

\[
h^X(t)X(t) + h^B B(t) = F_1(t, X(t))
\]

for all \( t \in [0, T_1] \).

Solution: let \( h^X(t) = \frac{\partial F_1}{\partial x}(t, X(t)) = F_{1,x}(t, X(t)) \).
**Γ-Hedging**

Introduce one more derivative: $F_2$ with $\Phi_2$ and $T_2 > T_1$. Form a hedge-portfolio $\{h^X, h^{F_2}, h^B\}$ and match the first and second derivatives w.r.t. $X$:

\[
F_1(t, X(t)) = h^X(t)X(t) + h^{F_2}(t)F_2(t, X(t)) + h^B(t)B(t),
\]
\[
\Delta^{F_1}(t, X(t)) = h^X(t) + h^{F_2}(t)\Delta^{F_2}(t, X(t)),
\]
\[
\Gamma^{F_1}(t, X(t)) = h^{F_2}(t)\Gamma^{F_2}(t, X(t)).
\]

This yields the portfolio

\[
h^X(t) = \Delta^{F_1}(t, X(t)) - \frac{\Gamma^{F_1}(t, X(t))}{\Gamma^{F_2}(t, X(t))}\Delta^{F_2}(t, X(t)),
\]
\[
h^{F_2}(t) = \frac{\Gamma^{F_1}(t, X(t))}{\Gamma^{F_2}(t, X(t))},
\]
\[
h^B(t) = \frac{F_1(t, X(t)) - h^X(t)X(t) - h^{F_2}(t)F_2(t, X(t))}{B(t)}.
\]
**Γ-Hedging**

Introduce one more derivative: $F_2$ with $\Phi_2$ and $T_2 > T_1$. Form a hedge-portfolio $\{h^X, h^{F_2}, h^B\}$ and match the first and second derivatives w.r.t. $X$:

\[
F_1(t, X(t)) = h^X(t)X(t) + h^{F_2}(t)F_2(t, X(t)) + h^B(t)B(t),
\]
\[
\Delta F_1(t, X(t)) = h^X(t) + h^{F_2}(t)\Delta F_2(t, X(t)),
\]
\[
\Gamma F_1(t, X(t))) = h^{F_2}(t)\Gamma F_2(t, X(t)).
\]

This yields the portfolio

\[
h^X(t) = \Delta F_1(t, X(t)) - \frac{\Gamma F_1(t, X(t))}{\Gamma F_2(t, X(t))}\Delta F_2(t, X(t)),
\]
\[
h^{F_2}(t) = \frac{\Gamma F_1(t, X(t))}{\Gamma F_2(t, X(t))},
\]
\[
h^B(t) = \frac{F_1(t, X(t)) - h^X(t)X(t) - h^{F_2}(t)F_2(t, X(t))}{B(t)}.
\]
Discrete Time Hedging

Since the portfolio processes in both the $\Delta$-hedging and the $\Gamma$-hedging case are continuous processes the hedge portfolio must be rebalanced at every time instant in order for the hedging error to equal zero.

- In practice this is not possible.
- Re-balance at an equidistant time grid, i.e. $t_i = i/n$.
- Let $\mathcal{R}(n)$ denote the hedging error using an equidistant time grid with $n$ re-balancing points. What properties of $\mathcal{R}(n)$ do we get?
Discrete Time Hedging

Since the portfolio processes in both the $\Delta$-hedging and the $\Gamma$-hedging case are continuous processes the hedge portfolio must be rebalanced at every time instant in order for the hedging error to equal zero.

- In practice this is not possible.
- Re-balance at an equidistant time grid, i.e. $t_i = i/n$.
- Let $R(n)$ denote the hedging error using an equidistant time grid with $n$ re-balancing points. What properties of $R(n)$ do we get?
Numerical experiment: $\Delta$-hedging

Model: Black and Scholes. Parameters: $s_0 = 100$, $K_1 = 100$, $K_2 = 120$, $T_1 = 0.5$, $T_2 = 1.5$, $r = 0.03$ and $\sigma = 0.2$.

Figure: $\Delta$-hedging. Blue line: $n = 10$, 
Numerical experiment: $\Delta$-hedging

Model: Black and Scholes. Parameters: $s_0 = 100, K_1 = 100, K_2 = 120, T_1 = 0.5, T_2 = 1.5, r = 0.03$ and $\sigma = 0.2$.

Figure: $\Delta$-hedging. Blue line: $n = 10$, green line: $n = 20$. 
Numerical experiment: $\Gamma$-hedging

**Figure:** $\Gamma$-hedging. Blue line: $n = 10$, 

---

**Introduction**

Setting Numerical Experiment Results Conclusions References

**Numerical experiment:** $\Gamma$-hedging

![Graph showing $\Gamma$-hedging with different weights and time frames.](image)

**Figure:** $\Gamma$-hedging. Blue line: $n = 10$, 

---

**Magnus Wiktorsson**

Higher Order Hedging Schemes

Bachelier June 23, 2010 9 / 17
Numerical experiment: $\Gamma$-hedging

**Figure:** $\Gamma$-hedging. **Blue line:** $n = 10$, **green line:** $n = 20$. 
Numerical experiment: order of convergence

Assume that: \( \mathbb{E}[\mathcal{R}^2(n)] = Cn^\alpha \) then
\[
\log_{10}(\mathbb{E}[\mathcal{R}^2(n)]) = \log_{10}(C) + \alpha \log_{10}(n).
\]

**Figure:** Squares (□): \( \Delta \)-hedging,
Numerical experiment: order of convergence

Assume that: \( \mathbb{E}[\mathcal{R}^2(n)] = Cn^\alpha \) then

\[
\log_{10}(\mathbb{E}[\mathcal{R}^2(n)]) = \log_{10}(C) + \alpha \log_{10}(n).
\]

Figure: Squares (\( \square \)): \( \Delta \)-hedging, circles (\( \circ \)): \( \Gamma \)-hedging.
Previous results

\(\Delta\)-Hedging

- Equidistant time grid, i.e. \(t_i = i/n\)
  - European options (Zhang, 1999): Order of convergence \(1/\sqrt{n}\), i.e. \(\lim_{n \to \infty} nE[\mathcal{R}^2(n)] = C\).
  - Digital options (Gobet and Temam, 2001): Order of convergence \(1/n^{1/4}\).

- Nonuniform time grid
  - Digital options (Geiss, 2002): Order of convergence \(1/\sqrt{n}\).

\(\Gamma\)-Hedging

- For the standard Black-Scholes model Gobet and Makhlouf (2009) gives non-sharp lower bounds for convergence rates for both equidistant and non-equidistant grids.
Results

Γ-hedging of an European option on an equidistant time grid (Brodén and Wiktorsson, 2009): Order of convergence $1/n^{3/4}$.

Recall that the assumptions A1-A3 are:

Let $\tilde{\sigma}(y) = \sigma(e^y)$.

**A1.**

(i) There is a positive constant $\sigma_0$ such that $\tilde{\sigma}(y) \geq \sigma_0$ for all $y \in \mathbb{R}$.

(ii) The function $\tilde{\sigma}$ is bounded, uniformly Lipschitz continuous in compact subsets of $\mathbb{R}$ and uniformly Hölder continuous.

**A2.** The functions $(\partial^k/\partial y^k)\tilde{\sigma}(y)$, $i \in \{1, 2, 3, 4\}$, are bounded.

**A3.** $\Phi_1(x) = (x - K_1)^+$, $\Phi_2(x) = (x - K_2)^+$ and $T_2 > T_1$. 


Results

Γ-hedging of an European option on an equidistant time grid (Brodén and Wiktorsson, 2009): Order of convergence $1/n^{3/4}$.

Theorem

If A1-A3 hold, then

$$\mathbb{E}[\mathcal{R}_\Gamma^2(n)] = n^{-3/2} T_1^{3/2} C_{3/2} \lim_{t \uparrow T_1} g(t) + o\left(n^{-3/2}\right)$$

$$= n^{-3/2} T_1^{3/2} C_{3/2} e^{-2rT_1} \frac{K_1^3 \sigma^3(K_1)}{4\sqrt{\pi}} P_{X(T_1)|X(0)=x_0}(K_1) + o\left(n^{-3/2}\right),$$

where

$$g(t) = (T_1 - t)^{3/2} \mathbb{E} \left[ e^{-2rt} F_{1,xxx}^2(t, X_t) X_t^6 \sigma^6(X_t) | X(0) = x_0 \right], C_{3/2} \approx 0.62881,$$

and $P_{X(T_1)|X(0)=x_0}(K_1)$ is $X$'s transition density.
Figure: Log mean squared error as a function of the log number of re-balancings $n$, for the Black and Scholes model. Parameters $K_1 = 100$, $T_1 = 0.5$, $s_0 = 100$, $r = 0.03$, $\sigma = 0.3$ and $N_{MC} = 10^5$. Dash-dotted line: estimate from the Theorem, MC estimate with: squares: $K_2 = 80$, triangles: $K_2 = 100$ and circles: $K_2 = 120$. 
Conclusions

- We have shown that when $\Gamma$-hedging a European option on an equidistant time grid the order of convergence is $1/n^{3/4}$.
- An explicit expression for the leading term of the second moment of the hedging error is derived.
- The expression serves as a good approximation of the real second moment of the hedging error also for $n < \infty$.

Further research

- Investigate higher order terms in the expansion of the hedging mean squared error in order to find an optimal choice of hedge instrument in a collection of possible hedge instruments.
- Hedging schemes using an arbitrary number of hedge instruments.
- More complicated market models.
References


Thanks for the attention!

Questions ??
Supplementary

\[ C_a = \sum_{k=1}^{\infty} \int_0^1 \int_0^x \int_0^w \frac{1}{(k - v)^a} \, dv \, dw \, dx = \int_0^\infty \frac{e^t - 1 - t - \frac{t^2}{2}}{\Gamma(a) t^{a+1} (e^t - 1)} \, dt. \]

which is well defined for \( 0 < a < 2 \).