Exotic Options in Multiple Priors Models

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Motivation

Mathematical Framework

Exotic Options in Multiple Priors Models

- Dual Expiry Options
- Piecewise monotone Payoffs

Conclusions and future Work
Motivation
American options:

- The right to buy or sell an underlying $S$ at any time prior to maturity $T$ subject to a contract
- Realizing the profit $A(t, (S_s)_{s \leq t})$ when exercised at $t$

Problem of the buyer:

- Exercise the option optimally choosing a strategy that maximizes the expected reward of the option, i.e. choose a stopping time $\tau^*$ that maximizes

$$
\mathbb{E}^P ((A(\tau, (S_s)_{s \leq \tau}))) \text{ over all stopping times } \tau \leq T
$$

under an appropriately chosen measure $P$
How to choose $P$?

- $P$ is the equivalent martingale measure in complete markets
- $P$ is the physical measure in real option models

Solution

- For fixed stochastic basis backward induction leads to the solution
- Snell envelope defines the value function of the problem through

$$U_T = A(T, (S_s)_{s \leq T}) / (1 + r)^T$$

$$U_t = \max \left\{ A(t, (S_s)_{s \leq t}) / (1 + r)^t, \mathbb{E}^P (U_{t+1} | \mathcal{F}_t) \right\}$$

for $t < T$

- Stop as soon as the value process reaches the payoff process
Motivation for Multiple Priors Models

- What is if the market is imperfect?
- Information is imprecise?
- Regulation imposes constraints on trading rules?

Several answers are possible:

- Superhedging
- Utility indifference pricing
- Risk measure pricing

Our approach:

- Ambiguity pricing
Aim of the paper

Ambiguity pricing

- Take the perspective of a decision maker who is uncertain about the underlying’s dynamics and uses a set of priors instead of a single one

  \[ \text{maximize} \quad \inf_{P \in \mathcal{P}} \mathbb{E}^P (A(\tau, S_\tau) / (1 + r)^\tau) \]

- Being pessimistic she maximizes the lowest expected return of option

- Concentrate on the effect of ambiguity and assume risk neutrality

- Model a consistent market under multiple priors assumption

- Study several exotic options of American style in the framework of ambiguity pricing

- Analyze the difference between classical expected return based pricing and the coherent risk pricing
Economically

- Ambiguity pricing leads to a valuation under a specific pricing measure.
- The pricing measure is rather a part of the solution than of the model itself.
- The pricing measure captures the fears of the decision maker and depends on the state and the payoff structure.

Mathematically

- The pricing measure might lose the independence property.
- Cut off rules are still optimal in this model.
- The use of the worst-case measure increases the complexity.
General Framework
The Mathematical Setup

- **A probability space** \((\Omega, \mathcal{F}, \mathbb{P}_0)\)

  - \(\Omega = \bigotimes_{t=1}^{T} \{0, 1\}\) – the set of sequences with values in \(\{0, 1\}\)
  - \(\mathcal{F}\) – the \(\sigma\)-field generated by all projections \(\epsilon_t : \Omega \rightarrow \{0, 1\}\)
  - \(\mathbb{P}_0\) – the uniform on \((\Omega, \mathcal{F})\)

- **A filtration** \((\mathcal{F}_t)_{t=0,\ldots,T}\) generated by the sequence \(\epsilon_1, \ldots, \epsilon_t\) with \(\mathcal{F}_t = \sigma(\epsilon_1, \ldots, \epsilon_t)\), \(\mathcal{F}_0 = \{\emptyset, \Omega\}\), \(\mathcal{F} = \mathcal{F}_T\)

![Binomial tree](image-url)

Figure 1: Binomial tree
A convex set of priors $\mathcal{P}$ defined via

$$\mathcal{P} = \{ P \in \Delta(\Omega, \mathcal{F}) | P(\epsilon_t = 1|\mathcal{F}_{t-1}) \in [p, \overline{p}] \forall t \leq T \}$$

for a fixed interval $[p, \overline{p}] \subset (0, 1)$

- $\mathcal{P}$ contains all product measures defined via $P_p(\epsilon_{t+1} = 1|\mathcal{F}_t) = p$ for a fixed $p \in [p, \overline{p}]$ and all $t \leq T$

- Denote by $\overline{P}$ the measure $P_{\overline{p}}$ and by $\underline{P}$ the measure $P_p$

- $\epsilon_1, \ldots, \epsilon_t$ are i.i.d under all product measures $P_p \in \mathcal{P}$

- In general, no independence
Lemma 1  *The above defined set of priors \( \mathcal{P} \) satisfies*

1. For all \( P \in \mathcal{P} \) \( P \sim \mathbb{P}_0 \)

- All measures in \( \mathcal{P} \) agree on the null sets
- We can identify \( \mathcal{P} \) with the set of density processes \( \mathcal{D} = \{ \mathcal{D}_t | t \leq T \} \) where

\[
\mathcal{D}_t = \left\{ \frac{dP}{d\mathbb{P}_0} \bigg|_{\mathcal{F}_t} \bigg| P \in \mathcal{P} \right\}
\]

- \( \inf \) is always a min
Lemma 2 \( \mathcal{P} \) is time-consistent in the following sense: Let \( P, Q \in \mathcal{P} \), \((p_t)_t, (q_t)_t \in (\mathcal{D}_t)_t\). For a fixed stopping time \( \tau \leq T \) define the measure \( R \) via

\[
    r_t = \begin{cases} 
        \frac{p_t}{q_t} & \text{if } t \leq \tau \\
        \frac{p_{\tau}q_t}{q_{\tau}} & \text{else}
    \end{cases}
\]

Then \( R \in \mathcal{P} \).

Time-consistency is equivalent to

- a version of The Law of Iterated Expectations
- fork-stability (FÖLLMER/SCHIED (2004))
- rectangularity (EPSTEIN/SCHNEIDER (2003))

\( \Rightarrow \) Allows to change the measure between periods
Ambiguous version of the Cox–Ross–Rubinstein model

- A market with 2 assets:
  - A riskless asset $B$ with interest rate $r > 0$
  - A risky asset $S$ evolving according to $S_0 = 1$ and

\[
S_{t+1} = \begin{cases} 
  S_t \cdot u & \text{if } \epsilon_{t+1} = 1 \\
  S_t \cdot d & \text{if } \epsilon_{t+1} = 0
\end{cases}
\]

- Assume $u \cdot d = 1$ and $0 < d < 1 + r < u$

- $\overline{P}/\underline{P}$ is the measure with the highest/lowest mean return

- Path-dependent increments

- Dynamical model adjustment without learning
Exercise problem of an ambiguity averse buyer

- For an option paying off $A(t, (S_s)_{s \leq t})$ when exercised at $t$:

- Choose a stopping time $\tau^*$ that maximizes

$$\min_{P \in \mathcal{P}} \mathbb{E}^P(A(\tau, (S_s)_{s \leq \tau})/(1 + r)^\tau)$$

over all stopping times $\tau \leq T$

- Compute

$$U_t^\mathcal{P} = \operatorname{esssup}_{\tau \geq t} \operatorname{essinf}_{P \in \mathcal{P}} \mathbb{E}^P(A(\tau, (S_s)_{s \leq \tau})/(1 + r)^\tau | \mathcal{F}_t)$$

– the ambiguity value of the claim at time $t$
Theorem 1 (RIEDEL (2009)) Given a set of measures $\mathcal{P}$ as above and a bounded payoff process $X$, $X_t = A(t, (S_s)_{s \leq t})/(1 + r)^t$, define the multiple priors Snell envelope $U^\mathcal{P}$ recursively by

$$U^\mathcal{P}_T = X_T$$

$$U^\mathcal{P}_t = \max\{X_t, \min_{P \in \mathcal{P}} \mathbb{E}^P (U^\mathcal{P}_{t+1} | \mathcal{F}_t)\} \text{ for } t < T$$

Then,

1. $U^\mathcal{P}$ is the value process of the multiple priors stopping problem for the payoff process $X$, i.e.

$$U^\mathcal{P}_t = \max_{\tau \geq t} \min_{P \in \mathcal{P}} \mathbb{E}^P (X_\tau | \mathcal{F}_t)$$

2. An optimal stopping rule is then given by

$$\tau^* = \inf\{t \geq 0 | U^\mathcal{P}_t = X_t\}$$
The Solution Method

**Duality result (Karatzas/ Kou (1998))**: There exists a $\hat{P} \in \mathcal{P}$ s.t.

$$U^P = U^{\hat{P}} \quad \mathbb{P}_0 \text{ - a.s.}$$

**To solve the problem**

- Identify the worst-case measure $\hat{P} \in \mathcal{P}$
- Refer to the classical solution

**Idea**

- Identify the worst-case measure for monotone claims
- Decompose more complicated claims in monotone parts
- Construct the worst-case measure pasting together the worst-case densities of the monotone parts
Exotic Options in Multiple priors Models
Multiple expiry options expiry at some date $\sigma < T$ in the future issuing a new option with conditions specified at $\sigma < T$

Often used as employee bonus and therefore are subject to trading restrictions

The value to the buyer/executive differs from the cost to the company of granting the option (Hall/Murphy (2002))

Multiple expiry feature causes a second source of uncertainty:
Shout options allow the buyer to shout and freeze the strike at-the-money at any time prior to maturity.

Can be seen as the option to abandon a project to conditions specified by the buyer.

There is uncertainty about the strike at time $0$ that is resolved at the time of shouting.

The payoff of the shout option at shouting is an at-the-money put of European style and the problem becomes:

$$\maximize A(\sigma, S_\sigma) = (S_\sigma - S_T)^+/(1 + r)^T$$

over all stopping times $\sigma \leq T$.

The task here is rather to start the process optimally than to stop it.
Since the payoff process is not adapted consider for $t \leq T$

\[
X_t = \text{essinf}_{P \in \mathcal{P}} \mathbb{E}^P ((S_t - S_T)^+/(1 + r)^T | \mathcal{F}_t) \\
= S_t \cdot g(t, \overline{P}) \\
= S_t \cdot (1 - \overline{p})^T \left( \sum_{k=0}^{k(t)} \binom{T - t}{k} \left( \frac{\overline{p}}{1 - \overline{p}} \right)^k (1 - d^{T - 2k}) \right) \\
\text{for } k(t) = \left\lfloor \frac{T - t}{2} \right\rfloor
\]

**Lemma 3** For all stopping times $\sigma \leq T$ we have

\[
\min_{P \in \mathcal{P}} \mathbb{E}^P (X_\sigma) = \min_{P \in \mathcal{P}} \mathbb{E}^P (A(\sigma, S_\sigma)/(1 + r)^T)
\]
We can maximize $X$ instead of the original payoff.

As a consequence we have

$$U_0^P = \text{essinf}_{P \in \mathcal{P}} \mathbb{E}^P \left( \text{essinf}_{Q \in \mathcal{P}} \mathbb{E}^Q((S_{\sigma^*} - S_T)^+ | \mathcal{F}_{\sigma^*}) \right)$$

$$= \min_{P \in \mathcal{P}} \mathbb{E}^P \left( S_{\sigma^*} \cdot g(\sigma^*, \overline{P}) \right)$$

$$= \mathbb{E}^P \left( S_{\sigma^*} \cdot g(\sigma^*, \overline{P}) \right)$$

where $\sigma^*$ is optimal.

The worst-case measure is defined by

$$\hat{P}(\epsilon_{t+1} | \mathcal{F}_t) = \begin{cases} \overline{p} & \text{if } \sigma^* < t \\ p & \text{else} \end{cases}$$
Lemma 4  *The optimal stopping time for the above problem is given by*

\[ \sigma^* = \inf\{ t \geq 0 : f(t) = x^* \} \]

where

\[ f(t) = g(t, \overline{P}) \cdot (\overline{p} \cdot u + (1 - \overline{p}) \cdot d)^t \]

*and* \( x^* \) *is the maximum of* \( f \) *on* \([0, T]\)

**Proof:** Generalized parking method and Optional Sampling

**Remarks**

1. Closed form solutions require exact study of the monotonicity of \( f \)

2. \( 1 - \overline{p} \geq (\overline{p} \cdot u + (1 - \overline{p}) \cdot d) \) is sufficient to have

\[ \sigma^* = 0 \]
U–shaped payoffs consist of two monotone parts allowing to benefit from change in the underlying independently of the direction of the change.

Often used as speculative instrument before important events.

Figure 2: Payoff of Straddle
- Up- and Down-movement can increase the value of the claim
- Uncertainty does not vanish over time

**Lemma 5** The value process is Markovian. For every $t \leq T$ the value function $v(t, \cdot)$ is quasi-convex and there exists a sequence $(\hat{x}_t)_{t \leq T}$ s.t. $v(t, \cdot)$ increases on $\{x_t > \hat{x}_t\}$ and decreases else.

**Proof:** Backward induction

- Proof uses explicitly the binomial structure of the model
- As a consequence we obtain

$$\hat{P}(\epsilon_{t+1} = 1|\mathcal{F}_t) = \begin{cases} \frac{p}{P} & \text{if } S_t < \hat{x}_t \\ \frac{p}{P} & \text{if } S_t \geq \hat{x}_t \end{cases}$$
The worst-case measure is mean-reverting (in a wider sense)

The drift changes every time $S$ hits a barrier and can happen arbitrary often

Fears of the decision maker are opposite to the market movement

Increments are not independent anymore

Idea: Use the generalized parking method again or upper and lower bounds
Conclusions
Conclusions

- A model of a multiple priors market provided
- A method to evaluate options in imperfect markets proposed
- Pricing measure for several classes of payoffs derived
- Worst-case measure is path-dependent in general
- The structure of the stopping times carries over in this model
- This is, however, due to the model and not a general result
Future work

Continuous-time analysis – Brownian motion setting

Infinite time modeling in continuous time

- Allows for closed form solutions and comparative statics
- Mathematical traps due to multiple measure structure
- More modeling necessary to build a meaningful mathematical model
Thank you for your attention!