Sample Path Large Deviations and Optimal Importance Sampling for Stochastic Volatility Models

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Research goal

• Find a robust method to improve Monte Carlo simulation performance when valuating path dependent options.
• Valid for stochastic volatility models.

To achieve this goal

• Use sample path Large Deviations Principles (LDP) to identify an asymptotically optimal importance sampling change of drift.
• Problem: standard LDP results do not apply to stochastic volatility models.
  • Volatility can degenerate, local Lipschitz condition violated.

Secondary research goal

• Prove LDP for common stochastic volatility models (e.g. Heston, Hull & White).
Talk Outline

Focus on path dependent option pricing.

- Problem setup.
- Review of Importance Sampling.
- Overview on constructing Asymptotically Optimal changes of drift.
  - Valid for general diffusions.
- Specification to Heston stochastic volatility model.
- Numerical example
  - Asian put option in Heston model.
Path Dependent Option Pricing

Setup:

- $S = \{S_t; 0 \leq t \leq T\}$ : Price process
- $G = G(S)$ : Path dependent option payoff


Primary example

- Heston model:

$$\frac{dS_t}{S_t} = rdt + \sqrt{v_t}dW_t$$
$$dv_t = \kappa (\theta - v_t) dt + \xi \sqrt{v_t}dB_t$$
$$d\langle W, B \rangle_t = \rho dt$$

- Asian put option:

$$G(S) = \left( K - \frac{1}{T} \int_0^T S_t dt \right)^+$$
Monte Carlo Simulation

To calculate $E_P [G]$, run a Monte Carlo simulation.

- Robust: only have to replicate price/volatility dynamics.

Problem: simulation inefficient if $G$ only pays off in rare events.

- $G \neq 0$ a “Large Deviation” from the norm.
- Asian put: $K \gg S_0$.

Estimating confidence intervals is difficult.

- Simulation variance artificially low.
Improving the Monte Carlo Simulation

Goal: Run an effective Monte Carlo simulation by using Importance Sampling.

- Change simulation measure from $P$ to $Q$ and change option payoff from $G$ to $G \frac{dP}{dQ}$ so that

$$E_Q \left[ G \frac{dP}{dQ} \right] = E_P [G]$$

Variance under $Q$:

$$\text{Var}_Q \left[ G \frac{dP}{dQ} \right] = E_P \left[ G^2 \frac{dP}{dQ} \right] - E_P [G]^2$$

Optimization problem: $\min_{Q \in A} E_P \left[ G^2 \frac{dP}{dQ} \right]$

- $A$ an appropriate family of equivalent measures.
Example

Arithmetic average Asian put option

\[ G(S) = \left( K - \frac{1}{T} \int_0^T S_t dt \right)^+ \]

in the Heston model when \( K \gg S_0 \).

Change of measure corresponds to two changes in drift:

- One for the volatility \( \nu \).
- One for the asset price \( S \).

Change the drift so option is more in the money.

- Compensate for change in drift by including the "scaling factor" in the option payoff.
Optimization Considerations

General optimization problem ill-posed: zero variance achieved for

\[ \frac{dQ}{dP} = \frac{G}{E_P[G]} \]

- Not allowable because \( E_P[G] \) unknown in the first place.

Questions:
- How to adjust notion of optimality?
- How to choose an appropriate family of measures \( \mathcal{A} \)?
- How to provide an optimal answer for a large class of functionals \( G \)?
Previous Work

Glasserman, Heidelberger, Shahabuddin (1999): use LDP to find an efficient change of measure.

- Work in Black-Scholes model. Partition $[0, T]$ to reduce to a finite dimensional problem.
- Approximate $E_P \left[ G^2 \frac{dP}{dQ} \right]$ by taking an asymptotic expansion as noise parameter goes away.
- Solve an associated minimization problem.


- Find an optimal continuous change of drift.
- Characterize optimal change of drift via an Euler-Lagrange equation, possibly with an explicit solution.
Asymptotic Optimality - General Idea

For now, consider the optimization problem:

$$\inf_{Q \in \mathcal{A}} E_P \left[ G(X)^2 \frac{dP}{dQ} \right]$$

$X$ is a $d$-dimensional diffusion satisfying

$$dX_t = b(X_t) dt + \sigma(X_t) dW_t; \quad X_0 = x$$

where $b : \mathbb{R}^d \mapsto \mathbb{R}^d$, $\sigma : \mathbb{R}^d \mapsto \mathbb{R}^{d \times d}$

Construct $\mathcal{A}$ by taking Cameron-Martin-Girsanov changes of measure:

$$\mathcal{A} = \left\{ P^h \left| \frac{dP^h}{dP} = \exp \left( \int_0^T u(h)_t dW_t - \frac{1}{2} \int_0^T \|u(h)_t\|^2 dt \right) , h \in \mathbb{H}^X_T \right\}$$

where

$$u(h)_t = \sigma^{-1}(h_t) \left( \dot{h}_t - b(h_t) \right)$$

$$\mathbb{H}^X_T = \left\{ h \mid h(0) = x, \int_0^T \|u(h)_t\|^2 dt < \infty \right\}$$
Asymptotic Optimality (2)

Imbed $X$ into the family of diffusions (for $0 < \varepsilon \leq 1$):

$$dX_t^\varepsilon = b(X_t^\varepsilon)dt + \sqrt{\varepsilon}\sigma(X_t^\varepsilon)dW_t; \quad X_0^\varepsilon = x$$

For $h \in H_T$ set

$$H^h(X, W) = 2 \log G(X) - \int_0^T u(h)'dW_t + \frac{1}{2} \int_0^T \|u(h)_t\|^2dt$$

With $W^\varepsilon = \sqrt{\varepsilon}W$

$$E_P \left[ G^2(X) \frac{dP}{dP^h} \right] = E_P \left[ \exp \left( \frac{1}{\varepsilon} H^h(X^\varepsilon, W^\varepsilon) \right) \right]$$

at $\varepsilon = 1$. The small noise approximation is

$$L(h) = \limsup_{\varepsilon \downarrow 0} \varepsilon \log E_P \left[ \exp \left( \frac{1}{\varepsilon} H^h(X^\varepsilon, W^\varepsilon) \right) \right]$$

$\hat{h}$ is asymptotically optimal if

$$\hat{h} = \arg\min_{\{h \in H_T\}} L(h)$$
Asymptotic Optimality and LDP

As $\varepsilon \downarrow 0$, $(X^\varepsilon, W^\varepsilon)$ “converges” to $(\phi_t, 0)$ where $\phi$ solves

$$\dot{\phi}_t = b(\phi_t), \quad \phi_0 = x$$

Sample path LDP identify precise rate of convergence for the law of $(X^\varepsilon, W^\varepsilon)$ to $\delta(\phi,0)$.

Classical result (Freidlin-Wentzell): for $H : C[0, T]^2 \mapsto \mathbb{R}$ bounded, continuous (supremum norm topology)

$$\lim_{\varepsilon \downarrow 0} \varepsilon \log E \left[ e^{-\frac{1}{\varepsilon} H(X^\varepsilon, W^\varepsilon)} \right] = -\inf_{\{(\phi, \psi) \in C[0, T]^2\}} (H(\phi, \psi) + I(\phi, \psi))$$

- Valid for $b, \sigma$ bounded, Lipschitz (some relaxation OK)
- Rate function:

$$I(\phi, \psi) = \begin{cases} \frac{1}{2} \int_0^T \|u(\phi)_t\|^2 dt & \phi \in \mathbb{H}_T, \psi = u(\phi) \\ \infty & \text{else} \end{cases}$$
Variational Considerations

Freidlin-Wentzell asymptotics imply

\[ L(h) = \sup_{\{\phi \in \mathbb{H}_T^X\}} \left( 2 \log G(\phi) + \frac{1}{2} \int_0^T \|u(h)_t - u(\phi)_t\|^2 dt - \int_0^T \|u(\phi)_t\|^2 dt \right) \]

Asymptotically optimal change of measure found by solving

\[ \inf_{\{h \in \mathbb{H}_T^X\}} \sup_{\{\phi \in \mathbb{H}_T^X\}} \left( 2 \log G(\phi) + \frac{1}{2} \int_0^T \|u(h)_t - u(\phi)_t\|^2 dt - \int_0^T \|u(\phi)_t\|^2 dt \right) \]

(1)

A lower bound:

\[ \sup_{\{\phi \in \mathbb{H}_T^X\}} \left( 2 \log G(\phi) - \int_0^T \|u(\phi)_t\|^2 dt \right) \]

(2)

Practical plan:

- Solve (2) and find maximizer \( \hat{\phi} \).
- With \( \hat{h} = \hat{\phi} \), see if \( L(\hat{h}) \) equals value in (2).
For any family $Q^\varepsilon$ of equivalent measures

$$
\liminf_{\varepsilon \downarrow 0} \varepsilon \log E_P \left[ G(X^\varepsilon)^{2/\varepsilon} \frac{dP}{dQ^\varepsilon} \right] \geq 2 \liminf_{\varepsilon \downarrow 0} \varepsilon \log E_P \left[ G(X^\varepsilon)^{1/\varepsilon} \right]
$$

$$
= \sup_{\{\phi \in H_X^T\}} \left( 2 \log G(\phi) - \int_0^T \|u(\phi)_t\|^2 dt \right)
$$

- If practical plan works, $\hat{h}$ is robust.

Consider when $X = W$. Euler-Lagrange equation for (2):

$$
D_\eta \left( 2 \log G(\phi) - \int_0^T \|\dot{\phi}_t\|^2 dt \right) = 0 \quad D_\eta : \text{Gâteaux derivative towards } \eta
$$

If $G$ is Fréchet differentiable, using a Taylor expansion

$$
E_{P_\hat{h}} \left[ G(W) \frac{dP}{dP_\hat{h}} \right] = G(\phi) \exp \left( -\frac{1}{2} \int_0^T \|\dot{\phi}_t\|^2 dt \right) E_{P_\hat{h}} \left[ \exp \left( R(W) \right) \right]
$$

where $R(W)$ contains no linear terms.

- Variance due to linear part of $\log(G)$ eliminated.
Application to Heston Model

In the Heston model, $X = (S, \nu)$, $W = (B, Z)$ and

$$ b(s, \nu) = \begin{pmatrix} rs \\ \kappa(\theta - \nu) \end{pmatrix} \quad \sigma(s, \nu) = \begin{pmatrix} \rho s \sqrt{\nu} & \tilde{\rho} s \sqrt{\nu} \\ \xi \sqrt{\nu} & 0 \end{pmatrix} $$

where $\tilde{\rho} = \sqrt{1 - \rho^2}$.

BIG PROBLEM : $\sigma$ is neither elliptic nor locally Lipschitz.

- Freidlin Wentzell LDP must be extended.

Fortunately:

- If $\nu$ satisfies a LDP by itself, then so does $(S, \nu)$. (R. (2010))
- $\nu$ satisfies LDP (Donati-Martin, Rouault, Yor, Zani (2004))
Application (2) - Questions

Does the Freidlin-Wentzell result apply to the unbounded and discontinuous function

\[ H^h(X, W) = 2 \log G(X) - \int_0^T u(h)' \, dW_t + \frac{1}{2} \int_0^T \| u(h)_t \|^2 \, dt \]

- Yes, if \( G \) bounded from above and \( h \) smooth enough.

\[ \int_0^T u(h)_t' \, dW_t = u(h)_T W_T - \int_0^T \dot{u}(h)_t' \, W_t \, dt \]

Do the variational problems in (1) and (2) admit maximizers?

- Yes, if \( G \) is continuous and bounded from above. (R. (2010))
  
  - Transfer problem to \( L^2[0, T] \) via \( u : H^{(S,v)}_T \hookrightarrow L^2[0, T] \).
  - \( u^{-1} \), \( G \) weakly continuous, functionals in (1), (2) coercive.
Numerical Example

For the Asian put option, the following parameter values are considered (Heston (1993))

\[ \kappa = 2, \theta = 0.09, \xi = 0.2, \nu_0 = 0.04, \]
\[ r = 0.05, T = 1, S_0 = 50, K = 30, \rho = -0.5. \]

Asymptotic Optimality holds for \( \hat{h} \) solving (2) with these values.

Optimal price drift
Numerical Example (2)

Optimal volatility drift

Interpretation:

- Under $P$ the option is out of the money.
- To bring the option into the money either
  - The “average” price path must come down.
  - The “average” volatility must go up.
Future Work

Run numerical simulations to see actual variance reduction.

- Black-Scholes model: $5X - 10X$ variance reduction typical. Does this carry over?

Apply methodology to options which depend more directly on volatility.

- Out of the money call or put: variance reduction obtained primarily by changing price drift.
- What about for a straddle option? No obvious direction to move the price.

Derive LDP for other stochastic volatility models.

- SABR, CEV
THANK YOU!

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