American-style options, stochastic volatility, and degenerate parabolic variational inequalities

Panagiota Daskalopoulos\textsuperscript{1} \quad Paul Feehan\textsuperscript{2}

\textsuperscript{1}Department of Mathematics  
Columbia University

\textsuperscript{2}Department of Mathematics  
Rutgers University

June 23, 2010  
6th World Congress  
Bachelier Finance Society, Toronto
Degenerate Markov processes and their associated parabolic PDEs are pervasive in finance.

Degenerate parabolic PDEs give rise to challenging terminal/boundary value problems (European-style options) and terminal/boundary value obstacle problems (American-style options).

What boundary conditions are appropriate or necessary?
Research goes back to Kohn and Nirenberg (1965).

A highly selective list includes Daskalopoulos and her collaborators, Feller, Freidlin, Koch, Kufner, Levendorskii, Opic, Pinsky, Stredulinsky, ...

Although previous research on degenerate elliptic/parabolic PDEs is extensive, more often than not, results often exclude even simple examples of interest in finance (CIR, Heston, etc).

Recent research due to Ekstrom and Tysk for CIR PDEs and Laurence and Salsa for solutions of American-style, multi-asset BSM option pricing problems.
Heston’s asset price process, $S(u) = \exp(X(u))$, is defined by

$$
\begin{align*}
    dX(u) &= (r - q - \frac{Y(u)}{2}) du + \sqrt{Y(u)} dW_1(u), \quad X(t) = x, \\
    dY(u) &= \kappa(\theta - Y(u)) du + \sigma \sqrt{Y(u)} dW_2(u), \quad Y(t) = y,
\end{align*}
$$

where $(W_1(u), dW_3(u))$ is two-dimensional Brownian motion, $W_2(u) := \rho W_1(u) + \sqrt{1 - \rho^2} W_3(u)$, $\kappa, \theta, \sigma$ are positive constants, $\rho \in (-1, 1)$, $r \geq 0$, $q \geq 0$, and $Y(u)$ is the variance process.
Option pricing problems for the Heston process lead to

- *Degenerate* parabolic differential equations,
- *Degenerate* parabolic variational inequalities,

for European and American-style option pricing problems, respectively.
Heston parabolic differential equation

If $-\infty \leq x_0 < x_1 < \infty$, let $\mathcal{O} := (x_0, x_1) \times (0, \infty)$ and $Q := [0, T) \times \mathcal{O}$. If $\psi : Q \rightarrow \mathbb{R}$ is a suitable function, for example, $\psi(t, x, y) = (K - e^x)^+ \text{ or } (e^x - K)^+$, and $r \geq 0$, define

$$u(t, x, y) := e^{-r(T-t)}\mathbb{E}_{Q}^{t,x,y} [\psi(T, X(T), Y(T))],$$

then we expect

$$-u' + Au = 0 \quad \text{on } Q, \quad u(T, \cdot) = \psi(T, \cdot) \quad \text{on } \mathcal{O},$$

where

$$-Au := \frac{y}{2} (u_{xx} + 2\rho \sigma u_{xy} + \sigma^2 u_{yy}) + (r - q - y/2)u_x + \kappa(\theta - y)u_y - ru.$$
Degenerate processes and degenerate parabolic PDEs
Elliptic variational inequalities for the Heston operator
Parabolic variational inequalities for the Heston operator

Degenerate elliptic or parabolic PDEs

Suppose \((t, x) \in Q = [0, T) \times \partial \) and \(\partial \subset \mathbb{R}^n\), and

\[-Au(t, x) := \frac{1}{2} \sum_{i,j} a_{ij}(t, x) \frac{\partial^2 u}{\partial x_i \partial x_j} (t, x)
+ \sum_i b_i(t, x) \frac{\partial u}{\partial x_i} (t, x) - c(t, x) u(t, x)\]

If \(\xi^T A(t, x) \xi \geq \mu(t, x) |\xi|^2\), \(\xi \in \mathbb{R}^n\), where \(\mu(x) > 0\), then \(A\) is elliptic (parabolic) on \(Q\) if \(\mu > 0\) on \(Q\), and \(A\) is uniformly elliptic (parabolic) on \(Q\) if \(\mu \geq \delta\) on \(Q\), for some constant \(\delta > 0\). This condition fails for the Heston operator, as \(\mu = 0\) along \(\{y = 0\}\) component of \(\partial \) and the operator is “degenerate”.

Daskalopoulos and Feehan
Stochastic volatility and degenerate variational inequalities
Weighted Sobolev spaces

**Definition**

We need a weight function when defining our Sobolev spaces,

\[
\omega(x, y) := \frac{2}{\sigma^2} y^{\beta - 1} e^{-\gamma|x| - \mu y}, \quad \beta = \frac{2\kappa\theta}{\sigma^2}, \quad \mu = \frac{2\kappa}{\sigma^2},
\]

for \((x, y) \in \mathcal{O}\) and a suitable positive constant, \(\gamma\). Then

\[
H^1(\mathcal{O}, \omega) := \{ u \in L^2(\mathcal{O}, \omega) : (1 + y)^{1/2} u \in L^2(\mathcal{O}, \omega),
\text{ and } y^{1/2} Du \in L^2(\mathcal{O}, \omega) \},
\]

where

\[
\| u \|_{H^1(\mathcal{O}, \omega)}^2 := \int_{\mathcal{O}} y (u_x^2 + u_y^2) \omega \, dx \, dy + \int_{\mathcal{O}} (1 + y) u^2 \omega \, dx \, dy.
\]
Let $H_0^1(\Omega, w)$ be the closure in $H^1(\Omega, w)$ of $C_c^1(\Omega) \cap H^1(\Omega, w)$. For $i = 0, 1$, let $H_0^1(\Omega \cup \Gamma_i, w)$ be the closure in $H^1(\Omega, w)$ of $C_c^1(\Omega \cup \Gamma_i) \cap H^1(\Omega, w)$, where

$$\Gamma_0 = (x_0, x_1) \times \{0\} \text{ and } \Gamma_1 = \{x_0, x_1\} \times (0, \infty),$$

and $\Gamma_1 = \{x_0\} \times (0, \infty)$ if $x_1 = +\infty$, $\Gamma_1 = \{x_1\} \times (0, \infty)$ if $x_0 = -\infty$, and $\Gamma_1 = \emptyset$ if $x_0 = -\infty$ and $x_1 = +\infty$. 
Gårding inequality

**Proposition**

Let \( q, r, \sigma, \kappa, \theta \in \mathbb{R} \) be constants such that

\[
\beta := \frac{2\kappa\theta}{\sigma^2} > 0, \quad \sigma \neq 0, \quad \text{and} \quad -1 < \rho < 1.
\]

Then for all \( u \in V \) such that \( u = 0 \) on \( \Gamma_1 \), where \( V = H^1(\partial, w) \),

\[
a(u, u) \geq \frac{1}{2} C_2 \| u \|_V^2 - C_3 \|(1 + y)^{1/2} u \|_{L^2(\partial, w)}^2.
\]
Continuity estimates

Proposition

Choose

- $\beta < 1$: $V = H^1(\mathcal{O}, \mathfrak{w})$ and $W = H^1_0(\mathcal{O} \cup \Gamma_1, \mathfrak{w})$;
- $\beta > 1$: $V = W = H^1(\mathcal{O}, \mathfrak{w})$.

Then

$$|a(u, v)| \leq C_1 \|u\|_V \|v\|_W, \quad \forall (u, v) \in V \times W,$$

where $C_1$ is a positive constant depending at most on the coefficients $r, q, \kappa, \theta, \rho, \sigma$. 
Elliptic variational inequality with (nonhomogeneous) Dirichlet boundary conditions

Let $f \in L^2(\mathcal{O}, \mathfrak{w})$ and $g, \psi \in H^1(\mathcal{O}, \mathfrak{w})$ such that $\psi \leq g$ on $\mathcal{O}$. For $\beta > 1$, find $u \in H^1(\mathcal{O}, \mathfrak{w})$ such that

$$a(u, v - u) \geq (f, v - u)_{L^2(\mathcal{O}, \mathfrak{w})}, \text{ with } u \geq \psi \text{ on } \mathcal{O} \text{ and } u = g \text{ on } \Gamma_1,$$

$$\forall v \in H^1(\mathcal{O}, \mathfrak{w}) \text{ with } v \geq \psi \text{ on } \mathcal{O} \text{ and } v = g \text{ on } \Gamma_1,$$

that is, $u - g, v - g \in H^1_0(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$. For $\beta < 1$, the statement is identical, except that the Dirichlet conditions are $u = g$ and $v = g$ on $\Gamma$, that is, $u - g, v - g \in H^1_0(\mathcal{O}, \mathfrak{w})$. 

Daskalopoulos and Feehan

Stochastic volatility and degenerate variational inequalities
Existence and uniqueness of solutions to the elliptic variational inequality

**Theorem**

*There exists a unique solution to the elliptic variational inequality for the Heston operator.*
Higher order regularity

Definition
Let

\[ H^2(\mathcal{O}, \mathfrak{m}) := \{ u \in L^2(\mathcal{O}, \mathfrak{m}) : (1+y)^{1/2} u, y^{1/2} Du, yD^2 u \in L^2(\mathcal{O}, \mathfrak{m}) \}, \]

where

\[ \| u \|_{H^2(\mathcal{O}, \mathfrak{m})}^2 := \int_{\mathcal{O}} \left[ y^2 \left( u^2_{xx} + 2u^2_{xy} + u^2_{yy} \right) \right. \]
\[ \left. + y \left( u^2_x + u^2_y \right) + (1+y)u^2 \right] \mathfrak{m} \, dx \, dy. \]

Let \( H^2_{\text{loc}}(\mathcal{O}, \mathfrak{m}) \) denote the space of functions \( u \in H^2(\mathcal{O}', \mathfrak{m}) \) for all \( \mathcal{O}' \subset \mathcal{O} \).
$H^2$ regularity for solutions to the elliptic Heston variational inequality

**Theorem**

Suppose $\psi(x, y) = (K - e^x)^+ \text{ or } (e^x - K)^+$. If $u$ is the solution to the Heston elliptic variational inequality, then $u \in H^2(\mathcal{O}, \mathcal{W})$. 

Daskalopoulos and Feehan

Stochastic volatility and degenerate variational inequalities
If $u \in H^2(\mathcal{O}, w)$ and $\psi \in H^1(\mathcal{O}, w)$, then the variational formulation has an equivalent *strong formulation* as a complementarity problem, which is to find $u \in V$ such that

$$Au - f \geq 0, \quad u - \psi \geq 0, \quad (Au - f)(u - \psi) = 0 \text{ on } \mathcal{O}.$$
Simple attempts to adapt the argument Bensoussan and Lions (1982) in their proof existence and uniqueness of solutions to the “strong” variational inequality to the Heston operator $A$ fail because the bilinear map defined by $A$ is non-coercive.
A change of dependent variable

To circumvent the lack of coerciveness, we employ the change of dependent variable

$$\tilde{u}(t, x, y) = e^{-\lambda(1+y)(T-t)}u(t, x, y), \quad u \in V, (t, x, y) \in Q,$$

by analogy with the familiar exponential shift change of dependent variable $\tilde{u} = e^{-\lambda(T-t)}u$. 
A change of dependent variable (continued)

- One finds that the non-coercive parabolic problem,

\[ -u' + Au = f \text{ on } Q, \quad u(T) = h \text{ on } \partial, \quad u = g \text{ on } \Sigma, \]

is transformed, for \( t \in [T - \delta, T] \) and sufficiently small \( \delta \), into an equivalent coercive parabolic problem,

\[ -\tilde{u}' + \tilde{A}\tilde{u} = \tilde{f} \text{ on } Q, \quad \tilde{u}(T) = h \text{ on } \partial, \quad \tilde{u} = \tilde{g} \text{ on } \Sigma, \]

- An obstacle condition \( u \geq \psi \) is transformed into an equivalent obstacle condition \( \tilde{u} \geq \tilde{\psi} \).
A change of dependent variable (continued)

The bilinear form on $V \times V$ (defined by the weight $w$) associated to the operator $\tilde{A}(t)$ (with suitable boundary conditions) is

$$\tilde{a}(t; \tilde{u}(t), v) := (\tilde{A}(t)\tilde{u}(t), v)_{L^2(\partial, w)}. \tag{1}$$

We then obtain the key continuity estimate and Gårding inequality for $\tilde{a}(t)$. 
Continuity estimate and Gårding inequality for the transformed Heston operator

Proposition

For a sufficiently large positive constant $\lambda$, depending only the coefficients of $A$, and a sufficiently small positive constant $\delta < T$, depending only on $\lambda$ and the coefficients of $A$, the bilinear map $\tilde{a}(t) : V \times V \to \mathbb{R}$ obeys

$$|\tilde{a}(t; u, v)| \leq C \|u\|_V \|v\|_V,$$

$$\tilde{a}(t; v, v) \geq \frac{\alpha}{2} \|v\|^2_V,$$

for all $u, v \in V$ and $t \in [T - \delta, T]$. 
Change of Sobolev weight and transformation back to original problem

The weight in our previous definition of weighted Sobolev spaces,

$$w(x, y) := \frac{2}{\sigma^2} y^{\beta-1} e^{-\gamma|x| - \mu y}, \quad (x, y) \in \mathcal{O},$$

is replaced, when transforming back from a solution $\tilde{u}$ to a solution $u$ to the original problem, by

$$\tilde{w}(x, y) := e^{-2\lambda M(1+y)} w(x, y)$$

$$= \frac{2}{\sigma^2} y^{\beta-1} e^{-\gamma|x| - \mu y - 2\delta \lambda(1+y)}, \quad (x, y) \in \mathcal{O},$$

where $M > T$ is a constant.
Setup for abstract parabolic equations and inequalities

Let $V$ be a reflexive Banach space with dual $V'$. Denote $V = L^2(0, T; V)$, with dual $V' = L^2(0, T; V')$. Let $H$ be a Hilbert space. The embeddings

$$V \hookrightarrow H \cong H' \hookrightarrow V',$$

are continuous, with $V \subset H$ dense. Let $\mathcal{A} : V \to V'$ be a continuous but not necessarily linear map. Typically, $\mathcal{A}(t, v) = \mathcal{A}(t)v(t)$, where $\mathcal{A}(t) : V \to V'$, $t \in [0, T]$. When $\mathcal{A}(t) \in \mathcal{L}(V, V')$, the transformed bilinear form $a(t) : V \times V \to \mathbb{R}$ is

$$a(t; u, v) := \mathcal{A}(t)u(v), \quad u, v \in V.$$

If $u \in D(\mathcal{A}(t)) = \{v \in V : \mathcal{A}(t)v \in H\}$, we write

$$(A(t)u, v)_H := \mathcal{A}(t)u(v), \quad v \in V.$$
Formulations of the Cauchy problem

Proposition (Showalter)

Let $u_0 \in H$ and $f \in \mathcal{V}'$. If $u \in \mathcal{V}$, the following are equivalent.

1. **(Strong)**

   $$ -u' + \mathcal{A} u = f \text{ in } \mathcal{V}', \quad u(T) = u_0. $$

2. **(Variational)** For each $v \in \mathcal{V} \cap W^{1,2}(0, T; H)$ with $v(0) = 0$,

   $$ \int_0^T \left[ (u, v') + \mathcal{A} u(v) - f(v) \right] \, dt - (u_0, v(T))_H = 0. $$

3. **(Weak)** For each $v \in \mathcal{V}$,

   $$ -\frac{d}{dt} (u, v)_H + \mathcal{A} u(v) = f(v) \text{ in } \mathcal{D}^*(0, T), \quad u(T) = u_0. $$
Existence and uniqueness for the abstract linear Cauchy problem

Proposition (Showalter)

Assume the operators $A(t)$ are in $\mathcal{L}(V, V')$ and that there is a constant $\alpha > 0$ such that

$$A(t)v(t) \geq \alpha \|v\|^2_V, \quad v \in V, \quad t \in [0, T].$$

Given $f \in \mathcal{V}'$, $u_0 \in H$, there is a unique solution $u \in \mathcal{V}$, $u' \in \mathcal{V}'$ to

$$-u' + A u = f \text{ in } \mathcal{V}', \quad u(T) = u_0.$$

and $u$ satisfies

$$\|u\|^2_V \leq (1/\alpha)^2 \left(\|f\|^2_{\mathcal{V}'}, + \|u_0\|^2_H\right).$$
One may use the penalization method as a stepping stone from existence (and regularity) for non-linear elliptic and parabolic equations to existence and regularity for solutions to elliptic and parabolic variational inequalities.

Existence and uniqueness for the Cauchy problem for the penalized parabolic equation follows from existence and uniqueness results for the non-linear abstract Cauchy problem (Showalter, 1997).
Existence and uniqueness for the abstract Cauchy problem for the penalized equation

Proposition

Let $A(t, \cdot) \in \mathcal{L}(V, V')$, $t \in [0, T]$ obey

1. The function $A(\cdot, v) : [0, T] \to V'$ is measurable, $\forall v \in V$.

2. There is a positive constant $\alpha$ such that

$$A(t, v)(v) \geq \alpha \|v\|^2_V, \quad t \in [0, T], v \in V.$$ 

Then, given $\psi \in \mathcal{H}$, $f \in \mathcal{V}'$, $u_0 \in H$ with $u_0 \geq \psi(T, \cdot)$, and $\varepsilon > 0$, there is a unique solution, $u_\varepsilon \in \mathcal{V}$, with $u'_\varepsilon \in \mathcal{V}'$, to

$$-u'_\varepsilon + A u_\varepsilon + \frac{1}{\varepsilon} (\psi - u_\varepsilon)^+ = f \text{ in } \mathcal{V}', \quad u_\varepsilon(T) = u_0 \text{ in } H.$$
Let $\mathcal{K} \subset \mathcal{V}$ be a convex subset. Given $f \in \mathcal{V}'$ and $u_0 \in H$, $u \in \mathcal{V}$ solves the strong problem if

$$u \in \mathcal{K}, \ u' \in \mathcal{V}'$$

$$- \int_0^T u'(v - u) \, dt + \mathcal{A} u(v - u) \geq f(v - u), \quad \forall v \in \mathcal{K},$$

$$u(T) = u_0.$$
Weak problem for a parabolic variational inequality

Given $f \in \mathcal{V}'$ and $u_0 \in H$, $u \in \mathcal{V}$ solves the weak problem if

$$u \in \mathcal{K},$$

$$- \int_0^T \nu'(v - u) \, dt + \mathcal{A}u(v - u) \geq f(v - u),$$

$$\forall v \in \mathcal{K} \text{ with } v' \in \mathcal{V}', \, v(T) = u_0.$$
Existence and uniqueness for the weak problem for a parabolic variational inequality

Suppose $K(t), t \in [0, T]$, is a non-decreasing family of closed, convex subsets of $V$ containing $u_0 \in H$. Then

$$\mathcal{K} = \{ v \in \mathcal{V} : v(t) \in K(t) \text{ a.e. } t \in [0, T] \}$$

is a closed and convex subset of $\mathcal{V}$. The next theorem is an application of results of Showalter on abstract parabolic variational inequalities in Banach spaces.
Existence and uniqueness for the weak problem for a parabolic variational inequality (continued)

**Theorem (Showalter)**

Suppose \( \mathcal{A}(t, \cdot) \in \mathcal{L}(V, V') \) are given with \( \mathcal{A}(t, v) \) measurable in \( t \in [0, T] \), \( \forall v \in V \), and

\[
\mathcal{A}(t, v)(v) \geq \alpha \| v \|_V^2, \quad \forall v \in V, t \in [0, T],
\]

for some \( \alpha > 0 \). Suppose \( K(t), t \in [0, T], \) is a non-decreasing family of closed, convex subsets of \( V \) containing \( u_0 \in H \). Then for each \( f \in \mathcal{V}' \) there is a unique solution \( u \in \mathcal{H} \) to

\[
\int_0^T (-v' + \mathcal{A} u - f)(v - u) \, dt \geq 0,
\]

\( \forall v \in \mathcal{H} \) with \( v' \in \mathcal{V}', v(T) = u_0 \).
Existence and uniqueness for the strong problem for a parabolic variational inequality

- Ultimately, we want a classical solution to the familiar “complementarity” formulation of the American-style option pricing problem.
- We can obtain such classical solutions by developing a regularity theory for solutions to the weak problem.
- It is more direct to adapt the Bensoussan-Lions approach using the Galerkin and penalization methods to establish existence and uniqueness for the strong problem for a parabolic variational inequality.
Existence and uniqueness for the \textit{strong} problem for a parabolic variational inequality (continued)

**Theorem**

Suppose $\mathcal{A}(t, \cdot) \in \mathcal{L}(V, V')$ are given with $\mathcal{A}(t, v)$ measurable in $t \in [0, T], \forall v \in V$, and, for some $\alpha > 0$,

$$\mathcal{A}(t, v)(v) \geq \alpha \|v\|^2_V, \quad \forall v \in V, t \in [0, T],$$

Let $\psi \in W^{1,2}(0, T; H)$, $\mathcal{K} = \{v \in V : v \geq \psi\}$, $u_0 \in \mathcal{K}$, and $f \in \mathcal{K}$. Then there is a unique solution $u \in \mathcal{K}$, $u' \in \mathcal{K}$ to

$$\int_0^T (-u' + \mathcal{A} u - f)(v - u) \, dt \geq 0, \quad \forall v \in \mathcal{K}, \quad u(T) = u_0.$$
Regularity for solutions to the strong problem for the parabolic Heston variational inequality

Using our weighted Sobolev spaces and estimates, we adapt the Bensoussan-Lions regularity theory to establish

**Theorem**

*In the situation of the existence and uniqueness theorem for the strong problem for a parabolic Heston variational inequality, suppose \( \psi(t, x, y) = (e^x - K)^+ \) or \( (K - e^x)^+ \). Then the solution \( u \) is in \( L^2(0, T; H^2(\mathcal{O}, w)) \).

Given this regularity, a solution to the strong problem for the parabolic Heston variational inequality is a solution to the more familiar complementarity form for the Heston variational inequality:
Theorem

Given $f \in L^2(0, T; L^2(Q, \mathcal{W}))$, $g \in L^2(0, T; H^2(\mathcal{O}, \mathcal{W}))$, $u_0(x, y) = \psi(t, x, y) = (e^x - K)^+ \text{ or } (K - e^x)^+$, then there is a unique $u \in L^2(0, T; H^2(\mathcal{O}, \mathcal{W}))$ solving

$$-u' + Au \geq f \text{ on } Q,$$

$$u \geq \psi \text{ on } Q,$$

$$(-u' + Au - f)(u - \psi) = 0 \text{ on } Q,$$

$$u = g \text{ on } (\Sigma_0 \cup \Sigma_1) \times [0, T) \ (if \ 0 < \beta < 1) \text{ or }$$

$$= g \text{ on } \Sigma_1 \times [0, T) \ (if \ \beta \geq 1),$$

$$u(T) = \psi \text{ on } \mathcal{O}.$$  

where $g \geq \psi \text{ on } \Sigma.$
Current research

- Global $W^{2,p}$ regularity.
- Regularity of the solution $u$ up to the boundary.
- Regularity of the free boundary separating the continuation and exercise regions.

REFERENCES

References (continued)