Control Improvement for Jump-Diffusion Processes with Applications to Finance

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Toronto, June 2010
Outline

- Motivation: MDPs
- Controlled Jump-Diffusion Processes
- Control Improvement Algorithm
- Financial Applications
Markov Decision Processes

Let \((X_n)\) be a controlled Markov process with
- state space \(S\), action space \(A\),
- transition kernel \(Q(\cdot | x, a)\).

Let \(f : S \rightarrow A\) be a decision rule and
- \(\beta \in (0, 1)\) a discount factor,
- \(r(x, a)\) a bounded reward function.

Consider the infinite-horizon Markov Decision Problem

\[
J(x) := \sup_{f \in F} J_f(x) = \sup_{f \in F} \mathbb{E}_x \left[ \sum_{n=0}^{\infty} \beta^n r(X_n, f(X_n)) \right].
\]
Notation

- \( B := \{ v : S \to \mathbb{R} : \|v\|_{\infty} < \infty \} \).
- For \( v \in B \) and \( f : S \to A \) let
  \[
  \mathcal{I}_f v(x) := r(x, f(x)) + \beta \int v(x') Q(dx'|x, f(x)).
  \]
- \( f^* \) is called maximizer of \( v \) if
  \[
  \mathcal{I}_{f^*} v = \sup_{f \in F} \mathcal{I}_f v.
  \]
- It holds that \( J_f = \mathcal{I}_f J_f \) and \( J = \sup_f \mathcal{I}_f J \).
Howard’s Policy Improvement Algorithm

1. Choose \( f_0 \) arbitrary and set \( k = 0 \).
2. Compute \( J_{f_k} \) as solution \( v \in B \) of the equation \( v = T_{f_k} v \).
3. Compute \( f_{k+1} \) as a maximizer of \( J_{f_k} \).
   Then \( J_{f_{k+1}} \geq J_{f_k} \). If \( f_{k+1} = f_k \) then \( J_{f_k} = J \) and \((f_k, f_k, \ldots)\) is optimal. Else set \( k := k + 1 \) and go to step 2.
Controlled Jump-Diffusion Processes

- $\mathbf{W} = (W_1, \ldots, W_m)$ is an $m$-dimensional Brownian motion,
- $\mathbf{N} = (N_1, \ldots, N_l)$ are indep. Poisson random measures,
- $\nu_j(B) := \mathbb{E} N_j(1, B)$ are the Lévy measures,
- $\tilde{N}_j(dt, dz_j) := N_j(dt, dz_j) - \nu_j(dz_j)dt$.

The $n$-dimensional controlled state process $X = (X_1, \ldots, X_n)$ is

\[
\begin{align*}
   dX_i(t) &= \mu_i(t, X_t, \pi_t)dt + \sum_{j=1}^{m} \sigma_{ij}(t, X_t, \pi_t)dW_j(t) + \\
   &+ \sum_{j=1}^{l} \int \gamma_{ij}(t, X_{t-}, \pi_{t-}, z_j)\tilde{N}_j(dt, dz_j)
\end{align*}
\]
Controlled Jump-Diffusion Processes

- $\pi = (\pi_t)$ is a càdlàg control process with values in $D \subset \mathbb{R}^d$,
- the coefficient functions $\mu, \sigma, \gamma$ are continuous,
- $g, h$ are reward functions.

Consider the problem

$$J^\pi(t, x) := \mathbb{E}_{t, x} \left[ \int_t^T g(s, X_s, \pi_s)ds + h(X_T) \right].$$

$$J(t, x) = \sup_{\pi} J^\pi(t, x).$$
Generator of the state process

\[ \mathcal{A} \nu(t, x, u) = \nu_t(t, x) + \sum_{i=1}^{n} \nu_{x_i}(t, x) \mu_i(t, x, u) + \]

\[ + \frac{1}{2} \sum_{i,j=1}^{n} (\sigma \sigma^T)_{ij}(t, x, u) \nu_{x_i x_j}(t, x) + \]

\[ + \sum_{j=1}^{l} \int \left( \nu(t, x + \gamma^{(j)}(t, x, u, z_j)) - \nu(t, x) - \nabla_x \nu(t, x) \gamma^{(j)}(t, x, u, z_j) \right) \nu_j(dz_j). \]
Control Improvement Algorithm

1. Suppose $\pi^0$ is an admissible control.
2. Compute the corresponding value function $J^0$ and suppose $J^0 \in C^{1,2}$.
3. Compute $\pi_1(t, x)$ such that it maximizes

$$u \mapsto g(t, x, u) + AJ^0(t, x, u), \quad u \in D$$

and suppose that $\pi^1_t := \pi_1(t, X^1_t)$ is an admissible control.
Control Improvement Algorithm

Under some technical conditions it holds:

**Theorem**

Let \( I := \{ (t, x) : g(t, x, \pi_1(t, x)) + AJ^0(t, x, \pi_1(t, x)) > 0 \} \).

a) If \( I \neq \emptyset \), then \( J^1(t, x) \geq J^0(t, x) \) for all \( (t, x) \) and \( J^1(t, x) > J^0(t, x) \) for \( (t, x) \in I \).

b) If \( I = \emptyset \) then \( \pi^1 \) is an optimal control.
Limit Considerations

Theorem

Suppose that the following assumptions are satisfied:

(i) \( \lim_{k \rightarrow \infty} J^k = J^\infty \in C^{1,2} \) and

\[
J_t^k \rightarrow J_t^\infty, \quad J_x^k \rightarrow J_x^\infty, \quad J_{xx}^k \rightarrow J_{xx}^\infty \quad \text{uniformly.}
\]

(ii) \( \mu, \sigma, \gamma \) are bounded.

Let \( \pi \) be a policy defined by the maximizer of \( J^\infty \) as in step (b) of the algorithm, then \( J = J^\infty \) and \( \pi \) is optimal.
Financial Market

- The price process \( (S_t^0) \) of the riskless bond is given by

\[
S_t^0 := e^{rt},
\]

where \( r \geq 0 \) denotes the fixed continuous interest rate.

- The price process \( (S_t) \) of the risky asset satisfies:

\[
dS_t = S_t \left( \mu dt + \sigma dW_t + \int_{-1}^{\infty} z \tilde{N}(dt, dz) \right)
\]

where \( \mu \in \mathbb{R}, \sigma > 0 \) and \( \int_{-1}^{\infty} z \nu(dz) < \infty \).

- Øksendal and Sulem (2005)
Portfolio Optimization

- $U : (0, \infty) \rightarrow \mathbb{R}$ is a (strictly increasing, concave) utility function.

- $(\pi_t)$ with $\pi_t \in [0, 1]$ is the portfolio strategy where $\pi_t = \text{fraction of wealth invested in the stock at time } t$.

The dynamics of the wealth process is

$$dX_t^\pi = X_t^\pi \left( r dt + \pi_t \cdot (\mu - r) dt + \pi_t \sigma dW_t + \pi_t \int_{-1}^{\infty} z \tilde{N}(dt, dz) \right).$$

The portfolio problem is

$$J(t, x) := \sup_{\pi} \mathbb{E}[U(X_T^\pi) | X_t^\pi = x].$$
When is the "invest all the money in the bond"-strategy optimal?

**Theorem**

Let $U \in C^2(0, \infty)$ be an arbitrary utility function. The "invest all the money in the bond"-strategy is optimal if and only if $\mu \leq r$. 
Control Improvement for Jump-Diffusion Processes

Application: Portfolio Optimization

Proof

Consider $\pi_t \equiv 0$ with $J^{\pi}(t, x) = U(xe^{r(T-t)})$.

$\pi^* \equiv 0$ is again a maximum point of $u \mapsto A J^{\pi}(t, x, u)$ on $[0, 1]$ if and only if

$$\frac{\partial}{\partial u} A J^{\pi}(t, x, u)|_{u=0} = (\mu - r)x J^{\pi}_{x} \leq 0.$$
Suppose now we have a Black-Scholes market. In case $\mu > r$, the first improvement of the "invest all the money in the bond"-strategy is given by

$$\pi_1(t, x) = -\frac{U'(xe^{r(T-t)})}{U''(xe^{r(T-t)})xe^{r(T-t)}} \cdot \frac{(\mu - r)}{\sigma^2}.$$ 

It relies on the Arrow-Pratt-Relative-Risk-Aversion Coefficient and the Merton-ratio. When the utility function is the power or logarithmic utility function, the first improvement yields already the optimal investment strategy.
When is a constant fraction optimal?

Suppose $\nu$ is concentrated on $(0, \infty)$, i.e. jumps are only upwards and that $2 \int x \nu(dx) < \mu - r$. Under these assumptions it holds:

**Theorem**

*The logarithm- and the power-utility are the only utility functions $U \in C^2(0, \infty)$ with $U \in C^2$ (up to a multiplicative constant) where the optimal portfolio invests a constant positive fraction of the wealth in the stock.*
Proof

$J^{\pi}$ and $\pi_t \equiv \pi$ are optimal if and only if $\pi$ is a maximum point of $u \mapsto AJ^{\pi}(t, x, u)$, $u \geq 0$, i.e.

$$(\mu - r)J^{\pi}_x + J^{\pi}_{xx}\sigma^2 x \pi + \int_0^\infty \left( J^{\pi}_x(t, x + \pi x z)z - J^{\pi}_x(t, x) \right) \nu(dz) = 0$$

and we must have $AJ^{\pi}(t, x, \pi) = 0$, i.e.

$$J^{\pi}_t + (r + (\mu - r)\pi)xJ^{\pi}_x + \frac{1}{2} J^{\pi}_{xx}\sigma^2 x^2 \pi^2 +$$

$$+ \int_0^\infty \left( J^{\pi}(t, x + \pi x z) - J^{\pi}(t, x) - J^{\pi}_x(t, x)\pi x z \right) \nu(dz) = 0.$$


Thank you very much for your attention!