A BSDE approach to Curve Following in Limit Order Markets

Felix Naujokat
(joint work with Nicholas Westray)

Humboldt Universität zu Berlin and Quantitative Products Laboratory

Bachelier Finance Society, 6th World Congress.
Toronto. June 24, 2010
We are given a target function and want to minimise the deviation of stock holdings to this function.

This is a classical problem in stochastic control and related to
- Tracking Brownian motion, e.g. Beneš, Shepp, and Witsenhausen (1980).
- Finite fuel problems, e.g. Karatzas (1985).
Applications in finance:
- Index tracking,
- Portfolio liquidation,
- Delta hedging,
- Trading at volume weighted average prices (VWAP).

There is a tradeoff between accuracy and cost.

We trade in a limit order market.
Diagram of a Limit Order Market

- **Price**: 99, 100, 101, 102
- **Volume**: 200, 400, 600, 800
- **Buy Side**
  - Best Ask Price
- **Sell Side**
  - Best Bid Price
Two Types of Orders

- The investor may submit a market order and consume volume in the book...

- ...or he may place a limit order and wait for execution.
The Minimisation Problem

- Given a control $u$, we assume that the stock holdings satisfy
  \[ dX^u(t) = u_1(t)N(dt) + u_2(t)dt, \quad X^u(0) = x. \]
- The investor wants to minimise the performance functional
  \[
  J(t, x, z, u) \triangleq \mathbb{E} \left[ \int_t^T g(u_2(s), Z(s)) + h(X^u(s) - \alpha(s, Z(s))) ds \\
  + f(X^u(T) - \alpha(T, Z(T))) \right]
  \]
  with cost function $g$, penalty functions $h$ and $f$, target function $\alpha$
  and a vector of stochastic signals $Z$, e.g. spread or index.
The Minimisation Problem ctd

We assume the following dynamics for the process $Z$:

$$dZ(t) = \mu(t, Z(t))dt + \sigma(t, Z(t))dW(t)$$

$$+ \int \gamma(t, Z(t), \theta) \tilde{M}(d\theta, dt), \quad Z(0) = z.$$  

The value function is defined as

$$\nu(t, x, z) \triangleq \inf_{u \in U} J(t, x, z, u).$$

Assumptions:

- Limit orders only on reference price, only full execution.
- $f, g$ and $h$ strictly convex, nonnegative, smooth and of quadratic growth
- $\alpha$ of polynomial growth, $\mu, \sigma$ and $\gamma$ Lipschitz.
Existence and Uniqueness

**Theorem (N. and Westray (2010) Theorem 3.1)**

There is a unique optimal control $\hat{u}$.

- The proof combines the following a priori estimate with a Komlos argument.

**Lemma**

1. There are constants $K_1 \in \mathbb{R}, K_2 > 0$ such that
   
   \[ J(t, x, z, u) \geq K_1 + K_2 \|u_2\|_{L^2}. \]

2. There is a constant $K_3 > 0$ such that if $\|u\|_{L^2} \geq K_3$ then $u$ cannot be optimal.
Lemma (Cadenillas (2002) Lemma 4.1)

The functional $J$ is Gâteaux differentiable.

- It is known that $\hat{u}$ is optimal iff $\langle J'(\hat{u}), u - \hat{u} \rangle \geq 0$ for all $u \in \mathcal{U}$.
- This yields the following characterisation in terms of the adjoint equation $(P, Q, R)$ (see next slide).

Theorem

A control $\hat{u}$ is optimal if and only if

1. $\hat{u}_2$ maximises $u_2 \mapsto g(u_2, z) - P(t)u_2$
2. $P(t-) + R_1(t) = 0$. 
The Coupled Forward-Backward System

- The adjoint equation is the following backward SDE

\[ dP(t) = h'(X^\hat{u}(t) - \alpha(t, Z(t))) \, dt + Q(t) \, dW(t) + R_1(t) \tilde{N}(dt) \]

\[ + \int_{\mathbb{R}^k} R_2(t, \theta) \tilde{M}(dt, d\theta), \]

\[ P(T) = -f'(X^\hat{u}(T) - \alpha(T, Z(T))). \]

- It is coupled with the forward SDE

\[ dX^\hat{u}(t) = \hat{u}_1(t) N(dt) + \hat{u}_2(t) dt, \]

\[ dZ(t) = \mu(t, Z(t)) dt + \sigma(t, Z(t)) dW(t) + \int_{\mathbb{R}^k} \gamma(t, Z(t-), \theta) \tilde{M}(dt, d\theta), \]

\[ X^\hat{u}(0) = x, Z(0) = z, \]

- via the optimality conditions

\[ \hat{u}_2(t, Z(t)) = \text{arg max}_{u_2} \{ g(u_2, Z(t)) - P(t)u_2 \} \text{ and } P(t-) + R_1(t) = 0. \]
We define the cost-adjusted target function as

\[ \tilde{\alpha}(t, z) = \arg \min_{x \in \mathbb{R}} v(t, x, z) \]

Analysing the FBSDE, we show that trading is directed towards \( \tilde{\alpha} \).

**Theorem**

1. The optimal limit order is \( u_1 = \tilde{\alpha}(t, z) - x \).
2. In the buy region \( \{x < \tilde{\alpha}\} \) we have \( u_1, u_2 > 0 \).
   In the sell region \( \{x > \tilde{\alpha}\} \) we have \( u_1, u_2 < 0 \).
   In the no trade region \( \{x = \tilde{\alpha}\} \) we have \( u_1, u_2 = 0 \).
Further analysis of the FBSDE yields that the map $\alpha \mapsto \tilde{\alpha}$ is monotone, translation invariant and bounded.

**Proposition**

- If $\alpha \geq \beta$ then $\tilde{\alpha} \geq \tilde{\beta}$.
- If $\beta = \alpha + K$ then $\tilde{\beta} = \tilde{\alpha} + K$ for any constant $K$.
- $\inf\alpha \leq \tilde{\alpha} \leq \sup\alpha$. 
For simple dynamics, we have closed form solutions.

**Proposition**

Let \( g(u_2, z) = \kappa u_2^2 \) and \( f(y) = h(y) = y^2 \) and

\[
dZ = \mu(t) dt + \sigma(t) dW(t).
\]

Then

\[
\tilde{\alpha}(t, z) = -\frac{b}{a}, \quad \hat{u}_1 = -\frac{b}{a} - x \quad \text{and} \quad \hat{u}_2 = -\frac{a}{2\kappa} \left(-\frac{b}{a} - x\right),
\]

where \( a \) and \( b \) solve some linear PDEs involving \( \alpha \) and are known explicitly.
We proved a version of the SMP and applied it to the problem of curve following in illiquid markets, allowing for limit and market orders.

We analysed the corresponding adjoint equation and derived the existence of buy and sell regions.

Explicit solution in special cases.

