Unified Multi-name Credit-Equity Modeling: A Multivariate Time Change Approach

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Joint work with: Vadim Linetsky
Introduction

- We develop a new class of multi-name unified credit-equity models that jointly model the stock prices of multiple firms, as well as their default events.
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  - We construct a multi-dimensional Markov semimartingale by applying a multivariate subordination of jump-to-default extended constant elasticity of variance (JDCEV) diffusions.

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- Some of the jumps are idiosyncratic to each firm, while some are either common to all firms (systematic), or common to a subgroup of firms.
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- We develop a new class of **multi-name unified credit-equity models** that jointly model the stock prices of multiple firms, as well as their default events,

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- Each of the stock prices experiences **state-dependent jumps with the leverage effect** (arrival rates of large jumps increase as the stock price falls), including the possibility of a **jump to zero** (jump to default).

- Some of the jumps are **idiosyncratic to each firm**, while some are either common to all firms (**systematic**), or common to a subgroup of firms.

- For the two-firm case, we obtain **analytical solutions for credit derivatives and equity derivatives**, such as basket options, in terms of eigenfunction expansions associated with the relevant subordinated semigroups.
Multi-name Credit-Equity Model Architecture

- We model the joint risk-neutral dynamics of stock prices $S^i_t$ of $n$ firms under an EMM $\mathbb{Q}$:

$$S^i_t = 1_{\{t < \tau_i\}} e^{\rho_i t} X^i_{\mathbb{T}^i_t} \equiv \begin{cases} 
    e^{\rho_i t} X^i_{\mathbb{T}^i_t}, & t < \tau_i \\
    0, & t \geq \tau_i \end{cases}, \quad i = 1, ..., n.$$ 

- Independent Diffusions $X^i$. 

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**Independent Diffusions $X^i_t$.**

- time-homogeneous, non-negative diffusion processes starting from positive values $X^i_0 = S^i_0 > 0$ (initial stock prices at time zero) and solving stochastic differential equations:

  $$dX^i_t = (\mu_i + k_i(X^i_t))X^i_t\, dt + \sigma_i(X^i_t)X^i_t\, dB^i_t$$
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  - $\sigma_i(x)$ is the state-dependent instantaneous volatility
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  - $\mu_i + k_i(x)$ is the state-dependent instantaneous drift, $\mu_i \in \mathbb{R}$ are constant parameters
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- $B^i$ are $n$ independent standard Brownian motions.
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- Multivariate Time Change $\mathcal{T}$.
  - $\mathcal{T}$ is an $n$-dimensional subordinator: A $n$-dimensional subordinator is a Lévy process in $\mathbb{R}^n_+ = [0, \infty)^n$ that is increasing in each of its coordinates.
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- Multivariate Time Change $T$.
  - $T$ is an $n$-dimensional subordinator: A \textit{n-dimensional subordinator} is a Lévy process in $\mathbb{R}_+^n = [0, \infty)^n$ that is increasing in each of its coordinates.
  - The (n-dimensional) Laplace transform of a n-dimensional subordinator is given by (here $u_i \geq 0$ and $\langle u, v \rangle = \sum_{i=1}^n u_i v_i$):

$$\mathbb{E}[e^{-\langle u, T_t \rangle}] = e^{-t \phi(u)}$$
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- The **Laplace exponent** given by the Lévy-Khintchine formula:

$$\phi(u) = \langle \gamma, u \rangle + \int_{\mathbb{R}^n_+} (1 - e^{-\langle u, s \rangle}) \nu(ds),$$
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  - The positive random variable $\tau_i$ models the time of default of the $i$th firm on its debt.
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  3. Define the $X_t^i$'s lifetime (we assume that $\inf\{\emptyset\} = H_0$ by convention):

$$\zeta_i := \inf\{t \in [0, H_0^i] : \int_0^t k_i(X_u^i)du \geq \mathcal{E}_i\}.$$
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  4. Then, time of default of the $i$th firm is defined by applying the time change $T^i$:

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      where $\nu_i$ is the Lévy measure of the one-dimensional subordinator $\tau^i$ ($\nu_i(A) = \nu(\mathbb{R}_+ \times \ldots \times A \times \ldots \mathbb{R}_+)$ with $A$ in the $i$th place, for any Borel set $A \subset \mathbb{R}_+$ bounded away from zero),
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    2. the constant $\rho_i$ is:
       $$\rho_i = r - q_i + \phi_i(-\mu_i),$$
       where $\phi_i(u)$ is the Laplace exponent of $T^i$,
       $\phi_i(u) = \phi(0, \ldots, 0, u, 0, \ldots, 0)$ ($u$ is in the $i$th place)
Credit-Equity Derivatives Pricing

- We are interested in pricing contingent claims written on multiple defaultable stocks.
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In particular, the price of a European-style derivative expiring at time $t > 0$ with the payoff function $f(S_t^1, ..., S_t^n)$ is given by

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- $S^i_t > 0$ (survival to time $t$, i.e., $\tau_i > t$) or,
- $S^i_t = 0$ (default by time $t$, i.e., $\tau_i \leq t$).
Multivariate Subordination of Multiparameter Semigroups

Thus we are interested on calculating expectations of the form

\[
\mathbb{E} \left[ 1_{\left\{ \tau_{1,2,\ldots,n} > t \right\}} f \left( X_{T_1}^1, X_{T_2}^2, \ldots, X_{T_n}^n \right) \right]
\]
Thus we are interested on calculating expectations of the form

$$E[1_{\{\tau_{1,2,\ldots,n} > t\}} f(X_{T_1}^1, X_{T_2}^2, \ldots, X_{T_n}^n)]$$

$$= E[1_{\{\tau_1 > t\}} \cdots 1_{\{\tau_n > t\}} f(X_{T_1}^1, X_{T_2}^2, \ldots, X_{T_n}^n)]$$

$$= E\left[1_{\{\tau_1 > t\}} \cdots 1_{\{\tau_n > t\}} f(X_{T_1}^1, X_{T_2}^2, \ldots, X_{T_n}^n) \right]$$

$$= \left( \tau_{1,\ldots,n} \right) = \Lambda_{i=1}^n \tau_i$$
Thus we are interested on calculating expectations of the form

\[
\mathbb{E}\left[\mathbf{1}_{\{\tau_{1,2,\ldots,n}>t\}} f\left(\frac{X_1^{\tau_1} T_1}{T}, \frac{X_2^{\tau_2} T_2}{T}, \ldots, \frac{X_n^{\tau_n} T_n}{T}\right)\right]
\]

\[
= \mathbb{E}\left[\mathbf{1}_{\{\tau_1>t\}} \cdots \mathbf{1}_{\{\tau_n>t\}} f\left(\frac{X_1^{\tau_1} T_1}{T}, \frac{X_2^{\tau_2} T_2}{T}, \ldots, \frac{X_n^{\tau_n} T_n}{T}\right)\right] \quad \left(\begin{array}{c}
\tau_{1,\ldots,n} = \bigwedge_{i=1}^n \tau_i \\
T_t \text{ & } X_t \text{ are indep.}
\end{array}\right)
\]

\[
= \mathbb{E}\left[\mathbb{E}\left[\mathbf{1}_{\{\zeta_1>T_1^t\}} \cdots \mathbf{1}_{\{\zeta_n>T_n^t\}} f\left(\frac{X_1^{\tau_1} T_1}{T}, \frac{X_2^{\tau_2} T_2}{T}, \ldots, \frac{X_n^{\tau_n} T_n}{T}\right) \mid T_t\right]\right] \quad \left(\begin{array}{c}
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Multivariate Subordination of Multiparameter Semigroups

Thus we are interested on calculating expectations of the form

\[ E \left[ \mathbf{1}_{\{\tau_1, \ldots, \tau_n > t\}} f \left( X^1_{T^1_t}, X^2_{T^2_t}, \ldots, X^n_{T^n_t} \right) \right] \]

\[ = E \left[ \mathbf{1}_{\{\tau_1 > t\}} \cdots \mathbf{1}_{\{\tau_n > t\}} f \left( X^1_{T^1_t}, X^2_{T^2_t}, \ldots, X^n_{T^n_t} \right) \right] \]

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Thus we are interested on calculating expectations of the form

$$
\mathbb{E}\left[1\{\tau_1,t,...,n > t\}\right] f\left(\frac{X_1^1}{T_t}, \frac{X_2^2}{T_t}, ..., \frac{X_n^n}{T_t}\right)
$$

$$
= \mathbb{E}\left[1\{\tau_1 > t\} \cdot \cdot \cdot 1\{\tau_n > t\}\right] f\left(\frac{X_1^1}{T_t}, \frac{X_2^2}{T_t}, ..., \frac{X_n^n}{T_t}\right)
$$

$$
= \mathbb{E}\left[\mathbb{E}\left[1\{\zeta_1 > T_1\} \cdot \cdot \cdot 1\{\zeta_n > T_n\}\right] f\left(\frac{X_1^1}{T_t}, \frac{X_2^2}{T_t}, ..., \frac{X_n^n}{T_t}\right) | T_t\right]
$$

$$
= \mathbb{E}\left[\mathbb{E}\left[1\{\zeta_1 > T_1\}\right] \cdot \cdot \cdot \mathbb{E}\left[1\{\zeta_n > T_n\}\right] f\left(\frac{X_1^1}{T_t}, \frac{X_2^2}{T_t}, ..., \frac{X_n^n}{T_t}\right) | T_t\right] \cdot \cdot \cdot | T_t\right]
$$

$$
= \mathbb{E}\left[\mathbb{E}\left[1\{\zeta_1 > T_1\}\right] \cdot \cdot \cdot \mathbb{E}\left[1\{\zeta_n > T_n\}\right] f\left(\frac{X_1^1}{T_t}, \frac{X_2^2}{T_t}, ..., \frac{X_n^n}{T_t}\right) | T_t\right] \cdot \cdot \cdot | T_t\right]
$$

$$
\left(\tau_1,...,n = \bigwedge_{i=1}^n \tau_i\right)
$$

$$
\left(T_t \& X_t \text{ are indep.}\right)
$$

$$
\left(\frac{X_t}{T_t} \text{ s are indep.}\right)
$$

(Psf)

Multivariant Subordination

Multi-parameter Semigroup
Thus we are interested on calculating expectations of the form

$$\mathbb{E}\left[ \mathbf{1}_{\{\tau_{1,2,\ldots,n} > t\}} f\left( X_{T_t}^1, X_{T_t}^2, \ldots, X_{T_t}^n \right) \right]$$

$$= \mathbb{E}\left[ \mathbf{1}_{\{\tau_1 > t\}} \cdots \mathbf{1}_{\{\tau_n > t\}} f\left( X_{T_t}^1, X_{T_t}^2, \ldots, X_{T_t}^n \right) \right]$$

$$= \mathbb{E}\left[ \mathbb{E}\left[ \mathbf{1}_{\{\zeta_1 > T_t^1\}} \cdots \mathbf{1}_{\{\zeta_n > T_t^n\}} f\left( X_{T_t}^1, X_{T_t}^2, \ldots, X_{T_t}^n \right) | T_t \right] \right]$$

$$= \mathbb{E}\left[ \mathbb{E}\left[ \mathbf{1}_{\{\zeta_1 > T_t^1\}} \cdots \mathbb{E}\left[ \mathbf{1}_{\{\zeta_n > T_t^n\}} f\left( X_{T_t}^1, X_{T_t}^2, \ldots, X_{T_t}^n \right) | T_t \right] \right] \cdots | T_t \right]$$

$$\int_{\mathbb{R}^n_+} (P_s f) \pi_t(ds)$$

\[\text{Multi-parameter Semigroup} \quad \text{Multi-subord. transition kernel}\]
Multivariate Subordination of Multiparameter Semigroups

Thus we are interested on calculating expectations of the form

\[
\mathbb{E}\left[\mathbf{1}_{\{\tau_1, \ldots, \tau_n > t\}} f\left(\frac{X_1}{T_1}, \frac{X_2}{T_2}, \ldots, \frac{X_n}{T_n}\right)\right]
\]

\[
= \mathbb{E}\left[\mathbf{1}_{\{\tau_1 > t\}} \cdots \mathbf{1}_{\{\tau_n > t\}} f\left(\frac{X_1}{T_1}, \frac{X_2}{T_2}, \ldots, \frac{X_n}{T_n}\right)\right]
\]

\[
= \mathbb{E}\left[\mathbb{E}\left[\mathbf{1}_{\{\zeta_1 > T_1\}} \cdots \mathbf{1}_{\{\zeta_n > T_n\}} f\left(\frac{X_1}{T_1}, \frac{X_2}{T_2}, \ldots, \frac{X_n}{T_n}\right) | T_t\right]\right]
\]

\[
= \mathbb{E}\left[\mathbb{E}\left[\mathbf{1}_{\{\zeta_1 > T_1\}} \cdots \mathbb{E}\left[\mathbf{1}_{\{\zeta_n > T_n\}} f\left(\frac{X_1}{T_1}, \frac{X_2}{T_2}, \ldots, \frac{X_n}{T_n}\right) | T_t\right] \cdots | T_t\right]\right]
\]

\[
= \int_{\mathbb{R}_+^n} \left(\mathcal{P}_s f\right) \pi_t(ds)
\]

Multivariate Subordination of Multiparameter Semigroups

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Default Correlation
Multivariate Subordination of Multiparameter Semigroups

Thus we are interested on calculating expectations of the form

\[ E[\textbf{1}_{\{\tau_1, \tau_2, \ldots, \tau_n > t\}} f(X_{1t}^{1}, X_{1t}^{2}, \ldots, X_{1t}^{n})] \]

\[ = E[\textbf{1}_{\{\tau_1 > t\}} \cdot \ldots \cdot \textbf{1}_{\{\tau_n > t\}} f(X_{1t}^{1}, X_{1t}^{2}, \ldots, X_{1t}^{n})] \]

\[ = E[E[\textbf{1}_{\{\zeta_1 > T_1^1\}} \cdot \ldots \cdot \textbf{1}_{\{\zeta_n > T_1^n\}} f(X_{1t}^{1}, X_{1t}^{2}, \ldots, X_{1t}^{n}) | T_t]] \]

\[ = E[E[\textbf{1}_{\{\zeta_1 > T_1^1\}}] \cdot \ldots \cdot E[\textbf{1}_{\{\zeta_n > T_1^n\}} f(X_{1t}^{1}, X_{1t}^{2}, \ldots, X_{1t}^{n}) | T_t] \cdot \ldots | T_t] \]

\[ = \int_{\mathbb{R}_+^n} (P^s f) \pi_t (ds) = \left( \tau^{\{1, \ldots, n\}} = \bigwedge_{i=1}^n \tau_i \right) \]

\[ = E[\textbf{1}_{\{\tau_1 > T_1^1\}} \cdot \ldots \cdot \textbf{1}_{\{\tau_n > T_1^n\}} f(X_{1t}^{1}, X_{1t}^{2}, \ldots, X_{1t}^{n})] \]

\[ = \left( T_t \& X_t \text{ are indep.} \right) \]

\[ = \left( \chi_t^{i'} \text{ are indep.} \right) \]

\[ \int_{\mathbb{R}_+^n} (P^s f) \pi_t (ds) = \left( \text{Subordinated Semigroup (one -- parameter)} \right) \]

Multivariate Subordination of Multiparameter Semigroups
Spectral Decomposition (I)

- We assume that all $X^i$ are 1D diffusions (symmetric Markov processes) on $(0, \infty)$ such that:

$$\text{the semigroups } P_i \text{ defined in the Hilbert spaces } H_i = L_2((0, \infty), m_i) \text{ endowed with the inner products } (f, g)_{m_i} = \int_0^\infty f(x)g(x)m_i(x)dx \text{ are symmetric with respect to the speed density } m_i(x), i.e., (P_i t f, g)_{m_i} = (f, P_i t g)_{m_i}, \forall t \geq 0, i = 1, \ldots, n.$$
Spectral Decomposition (I)

- We assume that all $X^i$ are 1D diffusions (symmetric Markov processes) on $(0, \infty)$ such that:

  - the semigroups $\mathcal{P}^i$ defined in the Hilbert spaces $\mathcal{H}_i = L^2((0, \infty), m_i)$ endowed with the inner products $(f, g)_{m_i} = \int_{(0, \infty)} f(x)g(x)m_i(x)dx$ are symmetric with respect to the speed density $m(x)$, i.e.,

$$\mathcal{P}^i_{t_i} f, g)_{m_i} = (f, \mathcal{P}^i_{t_i} g)_{m_i}, \quad \forall t_i \geq 0, \ & i = 1, \ldots, n$$
Spectral Decomposition (I)

We assume that all $X^i$ are 1D diffusions (symmetric Markov processes) on $(0, \infty)$ such that:

1. the semigroups $\mathcal{P}^i$ defined in the Hilbert spaces $\mathcal{H}_i = L^2((0, \infty), m_i)$ endowed with the inner products $(f, g)_{m_i} = \int_{(0, \infty)} f(x)g(x)m_i(x)dx$ are symmetric with respect to the speed density $m(x)$, i.e.,
   
   $$\left(\mathcal{P}^i_{t_i} f, g\right)_{m_i} = \left(f, \mathcal{P}^i_{t_i} g\right)_{m_i}, \quad \forall t_i \geq 0, \ & \ & i = 1, ..., n$$

2. Then $H = L^2((0, \infty)^n, m)$ is defined on the product space $(0, \infty)^n = (0, \infty) \times ... \times (0, \infty)$ with the product speed density $m(x) = m_1(x_1)\cdots m_n(x_n)$ and the inner product

   $$\left(f, g\right)_m = \int_{(0, \infty)^n} f(x)g(x)m(x)dx$$
Spectral Decomposition (II)

In the special case when each infinitesimal generator $G_i$ has a purely discrete spectrum with eigenvalues $\{-\lambda_k^i\}_{k=1}^{\infty}$ and the corresponding eigenfunctions $\varphi_k^i(x_i)$,

$$G_i \varphi_k^i(x_i) = -\lambda_k^i \varphi_k^i(x_i),$$
Spectral Decomposition (II)

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  \[ G_i \varphi_k^i(x_i) = -\lambda_k^i \varphi_k^i(x_i), \]

- the spectral representation of the multi-parameter semigroup takes the form of the eigenfunction expansion:

  \[ \mathcal{P}_t f = \sum_{k \in \mathbb{N}^n} e^{-\langle \lambda, t \rangle} c_k^f \varphi_k, \quad f \in H, \quad t = (t_1, \ldots, t_n) \geq 0, \]

  where $\sum_{k \in \mathbb{N}^n} = \sum_{k_1=1}^{\infty} \ldots \sum_{k_n=1}^{\infty}$, $\mathbb{N} = \{1, 2, \ldots\}$,
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$$P_t f = \sum_{k \in \mathbb{N}^n} e^{-\langle \lambda, t \rangle} c^f_k \varphi_k, \quad f \in H, \quad t = (t_1, ..., t_n) \geq 0,$$

where $\sum_{k \in \mathbb{N}^n} = \sum_{k_1=1}^{\infty} \cdots \sum_{k_n=1}^{\infty}$, $\mathbb{N} = \{1, 2, \ldots\}$,

the eigenvalues and eigenfunctions are

$$\lambda = (\lambda^1_{k_1}, ..., \lambda^n_{k_n})$$

$$\varphi_k(x) = \prod_{i=1}^{n} \varphi^i_{k_i}(x_i), \quad x_i \in (0, \infty), \quad x = (x_1, ..., x_n) \in (0, \infty)^n, \quad k \in \mathbb{N}^n,$$
Spectral Decomposition (II)

In the special case when each infinitesimal generator \( G_i \) has a purely discrete spectrum with eigenvalues \( \{-\lambda^i_k\}_{k=1}^{\infty} \) and the corresponding eigenfunctions \( \varphi^i_k(x_i) \),

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the eigenvalues and eigenfunctions are

\[
\lambda = (\lambda^1_{k_1}, \ldots, \lambda^n_{k_n})
\]

\[
\varphi_k(x) = \prod_{i=1}^{n} \varphi^i_{k_i}(x_i), \quad x_i \in (0, \infty), \quad x = (x_1, \ldots, x_n) \in (0, \infty)^n, \quad k \in \mathbb{N}^n,
\]

and the expansion coefficients are

\[
c^f_k = (f, \varphi_k)_m, \quad k \in \mathbb{N}^n.
\]
Spectral Decomposition of the Subordinated Semigroup $P_t^\phi$

Consequently, we can obtain the Spectral Decomposition of the Subordinated Semigroup as follows,

$$P_t^\phi f = \mathcal{E}^1 \left\{ \tau_1, 2, \ldots, n \right\} \mathcal{X}_2 T_2 t, \ldots, \mathcal{X}_n T_n t$$

Multivariate subordination of the $n$-parameter semigroup

Spectral representation of the $n$-parameter semigroup

Laplace transform of the $n$-dimensional subordinator

Levy-Khintchine exponent

Remark: When $n = 1$ the modeling framework is reduced to the Credit-Equity Model of Mendoza-Arriaga et al. (2009).
Spectral Decomposition of the Subordinated Semigroup $\mathcal{P}_t^\phi$

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$$\mathcal{P}_t^\phi f$$
Spectral Decomposition of the Subordinated Semigroup $\mathcal{P}_t^\phi$

Consequently, we can obtain the Spectral Decomposition of the Subordinated Semigroup as follows,

$$\mathcal{P}_t^\phi f = \mathbb{E}[\mathbf{1}_{\{\tau_{1,2,...,n}>t\}} f(X_{T_t}^1, X_{T_t}^2, ..., X_{T_t}^n)]$$
Spectral Decomposition of the Subordinated Semigroup $\mathcal{P}_t^\phi$

Consequently, we can obtain the Spectral Decomposition of the Subordinated Semigroup as follows,

$$
\mathcal{P}_t^\phi f = \mathbb{E} \left[ \mathbf{1}_{\{\tau_{1,2,\ldots,n} > t\}} f\left( X_{T_1}^1, X_{T_2}^2, \ldots, X_{T_n}^n \right) \right]
$$

$$
= \int_{\mathbb{R}_+^n} \mathcal{P}_s f \pi_t(ds)
$$

*(Multivariate subordination of the $n-$parameter semigroup)*
Spectral Decomposition of the Subordinated Semigroup $\mathcal{P}_t^\phi$

Consequently, we can obtain the Spectral Decomposition of the Subordinated Semigroup as follows,

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$$= \int_{\mathbb{R}^+} \mathcal{P}_s f \pi_t(ds)$$

$$= \int_{\mathbb{R}^n} \left( \sum_{k \in \mathbb{N}^n} e^{-\langle \lambda, s \rangle} c_k^f \varphi_k \right) \pi_t(ds)$$

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Consequently, we can obtain the Spectral Decomposition of the Subordinated Semigroup as follows,

$$
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$$

$$
= \int_{\mathbb{R}_+^n} \mathcal{P}_s f \pi_t(ds)
$$

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*Multivariate subordination of the $n$-parameter semigroup*

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$$= \int_{\mathbb{R}_+^n} \mathcal{P}_s f \pi_t(ds)$$

$$= \int_{\mathbb{R}_+^n} \left( \sum_{k \in \mathbb{N}^n} e^{-\langle \lambda, s \rangle} c_k^f \varphi_k \right) \pi_t(ds)$$

$$= \sum_{k \in \mathbb{N}^n} \left( \int_{\mathbb{R}_+^n} e^{-\langle \lambda, s \rangle} \pi_t(ds) \right) c_k^f \varphi_k$$

$$= \sum_{k \in \mathbb{N}^n} e^{-\phi(\lambda_{k1}^1, \ldots, \lambda_{kn}^n) t} c_k^f \varphi_k$$

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Spectral Decomposition of the Subordinated Semigroup $\mathcal{P}_t^\phi$

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$$= \int_{\mathbb{R}_+^n} \mathcal{P}_s f \pi_t(ds)$$

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$$= \sum_{k \in \mathbb{N}^n} e^{-\phi(\lambda_{k_1}^1, \ldots, \lambda_{k_n}^n) t} c_k^f \varphi_k$$

Remark: When $n = 1$ the modeling framework is reduced to the Credit-Equity Model of Mendoza-Arriaga et al. (2009).
Two Firms Illustration: \textit{the JDCEV process}

- Recall: we model the \textit{joint risk-neutral dynamics} of stock prices $S_t^i$ of 2 firms under an EMM $\mathbb{Q}$:

$$S_t^i = 1_{\{t < \tau_i\}} e^{\rho_i t} X_t^i T_t^i \equiv \begin{cases} e^{\rho_i t} X_t^i T_t^i, & t < \tau_i \vspace{1em} \\ 0, & t \geq \tau_i \end{cases}, \quad i = 1, 2$$
Two Firms Illustration: *the JDCEV process*

- **Recall:** we model the joint risk-neutral dynamics of stock prices $S_t^i$ of 2 firms under an EMM $\mathbb{Q}$:

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- Let $X_t^i$, $i = 1, 2$ be Jump-to-Default Extended Constant Elasticity of Variance (JDCEV) processes of Carr & Linetsky (2006):

  $$dX_t = [\mu + k(X_t)]X_t \, dt + \sigma(X_t)X_t \, dB_t, \quad X_0 = x > 0$$

  $$\sigma(X) = aX^\beta \quad \text{CEV Volatility} \quad \text{(Power function of } X)$$

  $$k(X) = b + c \sigma^2(X) \quad \text{Killing Rate} \quad \text{(Affine function of Variance)}$$

Rafael Mendoza

Default Correlation

McCombs
Two Firms Illustration: the JDCEV process

- **Recall:** we model the joint risk-neutral dynamics of stock prices $S_t^i$ of 2 firms under an EMM $Q$:

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CEV Volatility
(Power function of $X$)

$$k(X) = b + c \sigma^2(X)$$

Killing Rate
(Affine function of Variance)

- $a > 0 \Rightarrow$ volatility scale parameter (fixing ATM volatility)
- $\beta < 0 \Rightarrow$ volatility elasticity parameter
- $b \geq 0 \Rightarrow$ constant default intensity
- $c \geq 0 \Rightarrow$ sensitivity of the default intensity to variance
Two Firms Illustration: the JDCEV process

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- $b \geq 0$ \(\Rightarrow\) constant default intensity
- $c \geq 0$ \(\Rightarrow\) sensitivity of the default intensity to variance

For $c = 0$ and $b = 0$ the JDCEV reduces to the standard CEV process
Two Firms Illustration: *the JDCEV process*

- **Recall:** we model the joint risk-neutral dynamics of stock prices $S_t^i$ of 2 firms under an EMM $Q$:

$$S_t^i = 1_{\{t < \tau_i\}} e^{\rho_i t} X_{t}^i \equiv \begin{cases} e^{\rho_i t} X_{t}^i, & t < \tau_i \\ 0, & t \geq \tau_i \end{cases}, \quad i = 1, 2$$

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The model is consistent with:
leverage effect $\Rightarrow S \downarrow \rightarrow \sigma(S) \uparrow$
Two Firms Illustration: *the JDCEV process*

- **Recall:** we model the joint risk-neutral dynamics of stock prices $S^i_t$ of 2 firms under an EMM $\mathcal{Q}$:

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  $$ \sigma(X) = aX^\beta \quad \text{CEV Volatility} \quad (\text{Power function of } X) \quad \text{Killing Rate} \quad (\text{Affine function of Variance}) $$

  $$ k(X) = b + c \sigma^2(X) $$

  \(a > 0\) \quad \Rightarrow \text{volatility scale parameter (fixing ATM volatility)}

  \(\beta < 0\) \quad \Rightarrow \text{volatility elasticity parameter}

  \(b \geq 0\) \quad \Rightarrow \text{constant default intensity}

  \(c \geq 0\) \quad \Rightarrow \text{sensitivity of the default intensity to variance}

The model is consistent with:

- leverage effect $\Rightarrow S \downarrow \rightarrow \sigma(S) \uparrow$
- stock volatility–credit spreads linkage $\Rightarrow \sigma(S) \uparrow \leftrightarrow k(S) \uparrow$
JDCEV Eigenvalues and Eigenfunctions

- When $mu + b \neq 0$, the spectrum is purely discrete. When $mu + b < 0$, the eigenvalues and eigenfunctions are:

$$\lambda_n = \omega(n-1) + \lambda_1, \quad \varphi_n(x) = A^{\frac{\nu}{2}} \sqrt{\frac{(n-1)!|\mu + b|}{\Gamma(\nu + n)}} \times L_{n-1}^{\nu}(Ax^{-2\beta}), \quad n = 1, 2, \ldots,$$

where $L_n^{\nu}(x)$ are the generalized Laguerre polynomials.
When \( \mu + b \neq 0 \), the spectrum is purely discrete. When \( \mu + b < 0 \), the eigenvalues and eigenfunctions are:

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\]

where \( L_{n}^{\nu}(x) \) are the generalized Laguerre polynomials.

The principal eigenvalue \( \lambda_1 \), \( A \), \( \nu \) and \( \omega \) are,

\[
\lambda_1 := |\mu|, \quad A := \frac{|\mu + b|}{a^2|\beta|}, \quad \nu := \frac{1 + 2c}{2|\beta|}, \quad \omega := 2|\beta(\mu + b)|.
\]
When $mu + b \neq 0$, the spectrum is purely discrete. When $mu + b < 0$, the eigenvalues and eigenfunctions are:

$$
\lambda_n = \omega(n-1) + \lambda_1, \quad \varphi_n(x) = A^{\frac{\nu}{2}} \sqrt{\frac{(n-1)!|\mu + b|}{\Gamma(\nu + n)}} \times L^{\nu}_{n-1}(Ax^{-2\beta}), \quad n = 1, 2, \ldots,
$$

where $L^{\nu}_{n}(x)$ are the generalized Laguerre polynomials.

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\lambda_1 := |\mu|, \quad A := \frac{|\mu + b|}{a^2|\beta|}, \quad \nu := \frac{1 + 2c}{2|\beta|}, \quad \omega := 2|\beta(\mu + b)|,
$$

The speed density is defined as,

$$
m(x) = \frac{2}{a^2} x^{2c - 2 - 2\beta} e^{-Ax^{-2\beta}}
$$
Ex. Joint Survival Probability

Then the joint survival probability for two firms by time \( t > 0 \) is given by the eigenfunction expansion (\( x = (x_1, x_2) = (S_0^1, S_0^2) \)):

\[
Q(\tau_{\{1,2\}} > t) = \mathbb{E} \left[ 1_{\{\tau_{\{1,2\}} > t\}} \right]
\]
Then the joint survival probability for two firms by time $t > 0$ is given by the eigenfunction expansion ($x = (x_1, x_2) = (S^1_0, S^2_0)$):

$$Q(\tau_{\{1,2\}} > t) = \mathbb{E}\left[1_{\{\tau_{\{1,2\}} > t\}}\right]$$

$$= \sum_{n_1, n_2 = 1}^{\infty} e^{-\phi(\lambda^1_{n_1}, \lambda^2_{n_2}) t} c^1_{n_1} c^2_{n_2} \varphi^1_{n_1}(x_1) \varphi^2_{n_2}(x_2)$$
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Then the joint survival probability for two firms by time \( t > 0 \) is given by the eigenfunction expansion \((x = (x_1, x_2) = (S_0^1, S_0^2)):\)

\[
\mathbb{Q}(\tau_{\{1,2\}} > t) = \mathbb{E} \left[ \mathbf{1}_{\tau_{\{1,2\}} > t} \right]
\]

\[
= \sum_{n_1, n_2 = 1}^{\infty} e^{-\phi(\lambda_{n_1}^1, \lambda_{n_2}^2) t} c_{n_1} c_{n_2} \varphi_{n_1}^1(x_1) \varphi_{n_2}^2(x_2)
\]

Similarly, the single-name survival probabilities are given by the eigenfunction expansions:

\[
\mathbb{Q}(\tau_k > t) = \sum_{n=1}^{\infty} e^{-\phi_k(\lambda_n^k) t} c_n \varphi_n^k(x_k), \quad k = 1, 2.
\]
Ex. Joint Survival Probability

- Then the joint survival probability for two firms by time \( t > 0 \) is given by the eigenfunction expansion \((x = (x_1, x_2) = (S_0^1, S_0^2))\):

\[
Q(\tau_{1,2} > t) = \mathbb{E} \left[ \mathbf{1}_{\{\tau_{1,2} > t\}} \right] \\
= \sum_{n_1, n_2=1}^{\infty} e^{-\phi(\lambda_{n_1}^1, \lambda_{n_2}^2)t} c_{n_1}^1 c_{n_2}^2 \varphi_{n_1}^1(x_1) \varphi_{n_2}^2(x_2)
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\[
Q(\tau_k > t) = \sum_{n=1}^{\infty} e^{-\phi_k(\lambda_n^k)t} c_n^k \varphi_n^k(x_k), \quad k = 1, 2.
\]

The expansion coefficients are given by:

\[
c_n^k = (\varphi_n, 1)_m = \frac{1-2c_k}{4|\beta_k|} (1/(2|\beta_k|))_{n-1} \Gamma(c_k/|\beta_k| + 1) \\
\sqrt{(n-1)!|\mu_k + b_k| \Gamma(\nu_k + n)}
\]

where \((z)_n = z(z - 1)...(z - n - 1)\) is the Pochhammer symbol.
Ex. Joint Survival Probability

- Then the joint survival probability for two firms by time \( t > 0 \) is given by the eigenfunction expansion \((x = (x_1, x_2) = (S_0^1, S_0^2)):\)

\[
Q(\tau_{\{1,2\}} > t) = \mathbb{E} \left[ 1_{\{\tau_{\{1,2\}} > t\}} \right] = \sum_{n_1, n_2=1}^{\infty} e^{-\phi(x_1, x_2) t} c_{n_1} c_{n_2} \varphi_{n_1}(x_1) \varphi_{n_2}(x_2)
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- Similarly, the single-name survival probabilities are given by the eigenfunction expansions:

\[
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\]

\( \phi(u, v) \) is the Laplace exponent of the two-dimensional subordinator \((\mathcal{I}^1, \mathcal{I}^2)^\top\)
Ex. Joint Survival Probability

- Then the joint survival probability for two firms by time \( t > 0 \) is given by the eigenfunction expansion (\( x = (x_1, x_2) = (S_0^1, S_0^2) \)):

\[
Q(\tau_{\{1,2\}} > t) = \mathbb{E} \left[ \mathbf{1}_{\{\tau_{\{1,2\}} > t\}} \right]
\]

\[
= \sum_{n_1, n_2 = 1}^{\infty} e^{-\phi^1_n \cdot \phi^2_n} c^1_{n_1} c^2_{n_2} \varphi^1_{n_1}(x_1) \varphi^2_{n_2}(x_2)
\]

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\[
Q(\tau_k > t) = \sum_{n=1}^{\infty} e^{-\phi_k(\lambda_n)} c^n_k \varphi^n_k(x_k), \quad k = 1, 2.
\]

\( \phi(u, v) \) is the Laplace exponent of the two-dimensional subordinator \((T^1, T^2)^\top\)

\( \phi_1(u) := \phi(u, 0) \), and \( \phi_2(u) := \phi(0, u) \) are the Laplace exponents of the marginal one-dimensional subordinators \( T^k \), \( k \in \{1, 2\} \), respectively.
The default correlation has the form:

\[
\text{Corr}\left(\mathbf{1}_{\{\tau_1>t\}}, \mathbf{1}_{\{\tau_2>t\}}\right) = \frac{Q(\tau_{\{1,2}\}>t) - Q(\tau_1>t)Q(\tau_2>t)}{\prod_{k=1}^2 \sqrt{Q(\tau_k>t)(1-Q(\tau_k>t))}}
\]
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\text{Corr} \left( 1_{\{\tau_1 > t\}}, 1_{\{\tau_2 > t\}} \right) = \frac{Q(\tau_{\{1,2\}} > t) - Q(\tau_1 > t)Q(\tau_2 > t)}{\prod_{k=1}^2 \sqrt{Q(\tau_k > t)(1 - Q(\tau_k > t))}} \\
= \sum_{n \in \mathbb{N}_1^2} \left( e^{-\phi(\lambda_{n_1}^1, \lambda_{n_2}^2)t} - e^{-\left(\phi_1(\lambda_{n_1}^1) + \phi_2(\lambda_{n_2}^2)\right)t} \right) c_n \varphi_n(x) \\
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Default Correlation

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= \frac{\prod_{k=1}^{2} \sqrt{Q(\tau_k > t)(1-Q(\tau_k > t))}}{\prod_{k=1}^{2} \sqrt{Q(\tau_k > t)(1-Q(\tau_k > t))}} 
\]

- From this expression we observe that:

- it is zero if and only if \( \phi(u_1, u_2) = \phi(u_1, 0) + \phi(0, u_2), \)
The default correlation has the form:

\[
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\]

\[
= \sum_{n \in \mathbb{N}_1} \left( e^{-\phi(\lambda_{n_1}^1, \lambda_{n_2}^2)t} - e^{-\left(\phi(\lambda_{n_1}^1) + \phi(\lambda_{n_2}^2)\right)t} \right) c_n \varphi_n(x)
\]

From this expression we observe that:

- it is zero if and only if \( \phi(u_1, u_2) = \phi(u_1, 0) + \phi(0, u_2) \),

\[\Rightarrow \text{ That is, the coordinates } T^1 \text{ and } T^2 \text{ of the two-dimensional subordinator are independent.}\]
Consider a basket put option on the portfolio of two stocks with the payoff at time $t$

$$f(S^1_t, S^2_t) = (K - w^1 S^1_t - w^2 S^2_t)^+$$
Two Firms Basket Put Option

- Consider a basket put option on the portfolio of two stocks with the payoff at time $t$

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- We observe six contingent claims:
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We observe six contingent claims:

- One basket put that delivers the payoff if and only if both firms survive to maturity

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  $$1_{\{\tau_k > t\}} (K - w_k S_t^k)^+, \quad k = 1, 2$$

- An embedded multi-name credit derivative:
  $$K(1_{\{\tau_{\{1,2\}} > t\}} + 1 - 1_{\{\tau_1 > t\}} - 1_{\{\tau_2 > t\}}) = K 1_{\{\tau_1 \vee \tau_2 \leq t\}}$$
Two Firms Basket Put Option

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  \[ f(S^1_t, S^2_t) = (K - w_1 S^1_t - w_2 S^2_t)^+ \]

- We observe six contingent claims:
  - One basket put that delivers the payoff if and only if both firms survive to maturity
    \[ 1_{\{\tau_{\{1,2\}} > t\}}(K - w_1 S^1_t + w_2 S^2_t)^+ \]
  - Two single-name puts that deliver the payoffs if and only if both firms survive to maturity
    \[ 1_{\{\tau_{\{1,2\}} > t\}}(K - w_k S^k_t)^+ , \quad k = 1, 2 \]
  - Two single-name puts that deliver the payoffs if and only if the firm whose stock the put is written on survives to maturity.
    \[ 1_{\{\tau_k > t\}}(K - w_k S^k_t)^+ , \quad k = 1, 2 \]
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    \[ K(1_{\{\tau_{\{1,2\}} > t\}} + 1 - 1_{\{\tau_1 > t\}} - 1_{\{\tau_2 > t\}}) = K1_{\{\tau_1 \vee \tau_2 \leq t\}} \]

- We obtained explicit analytical solutions for all these claims.
Numerical Illustration

- We consider the two-name defaultable stock model.
Numerical Illustration

- We consider the two-name defaultable stock model.
- For this example the two diffusion processes $X$ are taken to be JDCEV with the same set of parameters are,

$$X_0 = x \quad a \quad b \quad c \quad q \quad \beta \quad \mu \quad r$$

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<td>0</td>
<td>-1</td>
<td>-0.3</td>
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</table>

**Table:** JDCEV parameter values.
Numerical Illustration

- We consider the two-name defaultable stock model.
- For this example the two diffusion processes $X$ are taken to be JDCEV with the same set of parameters are,

\[ X_0 = x \begin{array}{ccccccc} a & b & c & q & \beta & \mu & r \\ 50 & 10 & 0.01 & 0.5 & 0 & -1 & -0.3 & 0.05 \end{array} \]

Table: JDCEV parameter values.

- The volatility scale parameter $a$ in the local volatility function $\sigma(x) = ax^\beta$ is selected so that $\sigma(50) = 0.2$. 
Numerical Illustration

The two-dimensional subordinator $T$ is constructed from three independent Inverse Gaussian processes subordinators $S^i_t$, $i = 1, 2, 3$, as follows:

$$T^k_t = S^k_t + S^3_t, \quad k = 1, 2.$$  

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<tr>
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Table: IG parameter values.
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**Table:** IG parameter values.

In this specification $S^1_t$ and $S^2_t$ are two idiosyncratic components that influence only the first stock and the second stock, respectively, and
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The two-dimensional subordinator $T$ is constructed from three independent Inverse Gaussian processes subordinators $S^i_t$, $i = 1, 2, 3$, as follows:

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Table: IG parameter values.

In this specification $S^1$ and $S^2$ are two idiosyncratic components that influence only the first stock and the second stock, respectively, and $S^3_t$ is the systematic component common to both stocks.
Numerical Illustration

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$$T_t^k = S_t^k + S_t^3, \quad k = 1, 2.$$  

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- In this specification $S^1$ and $S^2$ are two idiosyncratic components that influence only the first stock and the second stock, respectively, and
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- The parameter $\eta$ is the decay parameter (damping parameter), which controls large size jumps $\Rightarrow S^3_t$ exhibits larger jumps.
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The parameter $\eta$ is the decay parameter (damping parameter), which controls large size jumps $S^3_t$ exhibits larger jumps.

Since the drift is zero ($\gamma = 0$) then the time changed processes $X^{i}_{T_t}$ are pure jump processes.
Numerical Illustration: Survival Probability

- As the stock price falls, the firm’s survival probability decreases

Figure: Single-name survival probability $Q(\tau > t)$ for $t = 1$ year as a function of the stock price $S_0 = x$. 
Numerical Illustration: Joint Survival Probability & Default Correlation

- As the stock prices fall, the joint survival probability also decreases which, in turn, causes the default correlation to decrease.

Figure: Joint survival probability $\mathbb{Q}(\tau \{1, 2\} > t)$ and default correlation $\text{Corr}(\mathbf{1}_{\{1 \tau_2 > t\}} , \mathbf{1}_{\{2 \tau_2 > t\}})$ for $t = 1$ year as functions of stock prices $S_0^1$ and $S_0^2$. 
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**Figure:** Joint survival probability $\mathbb{Q}(\tau_{\{1,2\}} > t)$ and default correlation $\text{Corr}(1_{\{\tau_1 > t\}}, 1_{\{\tau_2 > t\}})$ for $t = 1$ year as functions of stock prices $S_0^1$ and $S_0^2$.

- When the stock price is relatively high, the default can only be triggered by a large catastrophic jump to zero $\Rightarrow$ the systematic component $S^3$ governs large jumps.
Numerical Illustration: Joint Survival Probability & Default Correlation

- As the stock prices fall, the joint survival probability also decreases which, in turn, causes the default correlation to decrease.

![Figure](image-url)

(a) Joint survival probability. (b) Correlation of default indicators.

**Figure:** Joint survival probability $\mathbb{Q}(\tau_{\{1,2\}} > t)$ and default correlation $\text{Corr}(1_{\{\tau_1 > t\}}, 1_{\{\tau_2 > t\}})$ for $t = 1$ year as functions of stock prices $S^1_0$ and $S^2_0$.

- When the stock price is relatively high, the default can only be triggered by a large catastrophic jump to zero $\Rightarrow$ the systematic component $S^3$ governs large jumps.
- When the stock price is low, a smaller jump is enough to trigger default $\Rightarrow$ the idiosyncratic components $S^1$ and $S^2$ primarily govern small jumps.
Numerical Illustration: Joint Survival Probability & Default Correlation

- The price of a European-style basket put option on the equally-weighted portfolio of two stocks \((w_1 = w_2 = 1)\) with one year to maturity \((t = 1)\) and with the strike price \(K = 100\) as a function of the initial stock prices \(S_{10}^1\) and \(S_{20}^2\).

**Figure:** Two-name basket put prices for the range of initial stock prices \(S_{10}^1\) and \(S_{20}^2\) from zero to $60 for one year time to maturity and \(K = 100\).
Numerical Illustration: Joint Survival Probability & Default Correlation

- The price of a European-style basket put option on the equally-weighted portfolio of two stocks ($w_1 = w_2 = 1$) with one year to maturity ($t = 1$) and with the strike price $K = 100$ as a function of the initial stock prices $S_{01}$ and $S_{02}$.

**Figure:** Two-name basket put prices for the range of initial stock prices $S_{01}$ and $S_{02}$ from zero to $60$ for one year time to maturity and $K = 100$.

- When both firms are in default, $(S_{01}, S_{02}) = (0, 0)$, the price of the basket put is equal to the discounted strike $K$. 
Numerical Illustration: Joint Survival Probability & Default Correlation

- The price of a European-style basket put option on the equally-weighted portfolio of two stocks ($w_1 = w_2 = 1$) with one year to maturity ($t = 1$) and with the strike price $K = 100$ as a function of the initial stock prices $S^{1}_0$ and $S^{2}_0$.

Figure: Two-name basket put prices for the range of initial stock prices $S^{1}_0$ and $S^{2}_0$ from zero to $60$ for one year time to maturity and $K = 100$.

- When both firms are in default, $(S^{1}_0, S^{2}_0) = (0, 0)$, the price of the basket put is equal to the discounted strike $K$.
- When one of the two firms is in default, the basket put reduces to the single-name European-style put on the surviving stock with the strike $K$. 
Conclusion

- We propose a modeling framework based on multi-variate subordination of diffusion processes.
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- We propose a modeling framework based on multi-variate subordination of diffusion processes.

1. We start with \( n \) independent jump-to-default extended diffusions for \( n \) stocks.
2. Then we time change each one with a coordinate of a \( n \)-dimensional Subordinator.
Conclusion

- We propose a modeling framework based on multi-variate subordination of diffusion processes.

1. We start with *n independent jump-to-default extended diffusions* for *n* stocks.
2. Then we time change each one with a coordinate of a *n*-dimensional Subordinator

⇒ the result is *multi-name credit-equity model* with dependent jumps and jumps-to-default for all stocks.
Conclusion

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\[ \Rightarrow \text{the result is multi-name credit-equity model with dependent jumps and jumps-to-default for all stocks.} \]

- The dependence among jumps is governed by the Lévy measure of the \( n \)-dimensional subordinator.
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2. Then we time change each one with a coordinate of a \( n \)-dimensional Subordinator

\[ \Rightarrow \] the result is multi-name credit-equity model with dependent jumps and jumps-to-default for all stocks.

- The dependence among jumps is governed by the Lévy measure of the \( n \)-dimensional subordinator.

- The semigroup theory provides powerful analytical and computational tools for securities pricing.
Conclusion

- We propose a modeling framework based on multi-variate subordination of diffusion processes.
  
  1. We start with \( n \) independent jump-to-default extended diffusions for \( n \) stocks.
  2. Then we time change each one with a coordinate of a \( n \)-dimensional Subordinator

  \[\Rightarrow\] the result is \textit{multi-name credit-equity model} with dependent jumps and jumps-to-default for all stocks.

- The dependence among jumps is governed by the Lévy measure of the \( n \)-dimensional subordinator.

- The semigroup theory provides powerful analytical and computational tools for securities pricing.

- Thank you!
If \( \{ \mathcal{P}_t, t \in \mathbb{R}_n^+ \} \) is a \( n \)-parameter strongly continuous semigroup on a Banach space \( B \), then:

- It is the product of \( n \) one-parameter strongly continuous semigroups \( \{ \mathcal{P}_i t, t \geq 0 \} \) on \( B \) with infinitesimal generators \( G_i \) with domains \( \text{Dom}(G_i) \subset B \).
- That is, for \( t = (t_1, \ldots, t_n) \) we have: \( \mathcal{P}_t = \prod_{i=1}^{n} \mathcal{P}_i t_i \) and the semigroup operators \( \mathcal{P}_i t_i \) commute with each other, \( t_i \geq 0 \), \( i = 1, \ldots, n \).

In our case, the expectation operators associated with the Markov processes \( X_i \) define the corresponding semigroups \( \{ \mathcal{P}_i t, t \geq 0 \} \), \( \mathcal{P}_i t f(x_i) := E_{x_i} [\mathbb{1}_{\{\zeta_i > t_i\}} f(X_i t_i)] \), \( x_i \in E_i \), \( t_i \geq 0 \), in Banach spaces of bounded Borel measurable functions on \( E_i \).
Multiparameter Semigroup

If \( \{ \mathcal{P}_t, t \in \mathbb{R}_+^n \} \) is a \( n \)-parameter strongly continuous semigroup on a Banach space \( B \), then:

\[ \Rightarrow \text{it is the product of } n \text{ one-parameter strongly continuous semigroups } \{ \mathcal{P}^i_t, t \geq 0 \} \text{ on } B \text{ with infinitesimal generators } G_i \text{ with domains } \text{Dom}(G_i) \subset B. \]
If \{P_t, t \in \mathbb{R}^n_+\} is a $n$-parameter strongly continuous semigroup on a Banach space $B$, then:

\[ \Rightarrow \] it is the product of $n$ one-parameter strongly continuous semigroups \{P^i_t, t \geq 0\} on $B$ with infinitesimal generators $G_i$ with domains $\text{Dom}(G_i) \subset B$.

That is, for $t = (t_1, ..., t_n)$ we have:

\[ P_t = \prod_{i=1}^{n} P^i_{t_i} \]

and the semigroup operators $P^i_{t_i}$ commute with each other, $t_i \geq 0$, $i = 1, ..., n$. 
Multiparameter Semigroup

If \( \{P_t, t \in \mathbb{R}^n_+\} \) is a \( n \)-parameter strongly continuous semigroup on a Banach space \( B \), then:

\[ \Rightarrow \text{it is the product of } n \text{ one-parameter strongly continuous semigroups} \{P^i_t, t \geq 0\} \text{ on } B \text{ with infinitesimal generators } G_i \text{ with domains } \text{Dom}(G_i) \subset B. \]

- That is, for \( t = (t_1, ..., t_n) \) we have:

\[ P_t = \prod_{i=1}^{n} P^i_{t_i} \]

and the semigroup operators \( P^i_{t_i} \) commute with each other, \( t_i \geq 0, \ i = 1, ..., n. \)

- In our case, the expectation operators associated with the Markov processes \( X^i \) define the corresponding semigroups \( \{P^i_{t_i}, t_i \geq 0\}, \)

\[ P^i_{t_i} f(x_i) := \mathbb{E}_{x_i} [\mathbf{1}_{\{\xi_i > t_i\}} f(X^i_{t_i})], \quad x_i \in E_i, \quad t_i \geq 0, \]

in Banach spaces of bounded Borel measurable functions on \( E_i. \)
The embedded multi-name credit derivative with the notional amount equal to the strike price $K$ and paid at maturity if both firms default

$$e^{-rt}E[K1_{\tau_1 \vee \tau_2 \leq t}] = e^{-rt}K(1 + \mathbb{Q}(\tau_{\{1,2\}} > t) - \mathbb{Q}(\tau_1 > t) - \mathbb{Q}(\tau_2 > t))$$

where the joint survival probability $\mathbb{Q}(\tau_{\{1,2\}} > t)$ and marginal survival probabilities $\mathbb{Q}(\tau_k > t), k = 1, 2$; were given earlier.
The basket put that delivers the payoff if and only if both firms survive to maturity

\[ e^{-rt} \mathbb{E}\left[1_{\{\tau_{1,2} > t\}} (K - w_1 S^1_t + w_2 S^2_t)^+ \right] \]
Two Firms Basket Put Option

- The basket put that delivers the payoff if and only if both firms survive to maturity

\[
e^{-rt} \mathbb{E} \left[ 1_{\{\tau_{1,2} > t\}} (K - w_1 S^1_t + w_2 S^2_t)^+ \right]
\]

\[
e^{-r t} \sum_{n_1, n_2 = 1}^{\infty} e^{-\phi(\lambda_{n_1}^1, \lambda_{n_2}^2) t} c_{n_1, n_2}(K) \varphi_{n_1}(x_1) \varphi_{n_1}(x_2)
\]

Where the expansion coefficient \( c_{n_1, n_2}(K) \) is given by,
The basket put that delivers the payoff if and only if both firms survive to maturity

\[ e^{-rt} \mathbb{E}\left[1_{\{\tau_{1,2}>t\}}(K - w_1 S^1_t + w_2 S^2_t)^+\right] \]

\[ = e^{-rt} \sum_{n_1, n_2=1}^{\infty} e^{-\phi(\lambda^1_{n_1}, \lambda^2_{n_2}) t} c_{n_1, n_2}(K) \phi^1_{n_1}(x_1) \phi^2_{n_1}(x_2) \]

Where the expansion coefficient \( c_{n_1, n_2}(K) \) is given by,

\[ c_{n_1, n_2}(K) = \left( (K - w_1 x_1 - w_2 x_2)^+, \varphi_n(x) \right)_m \]

\[ = \int_{\mathbb{R}^2^+} (K - w_1 x_1 - w_2 x_2)^+ \phi^1_{n_1}(x_1) \phi^2_{n_2}(x_2) m_1(x_1) m_2(x_2) dx_1 dx_2 \]
Two Firms Basket Put Option

- The basket put that delivers the payoff if and only if both firms survive to maturity

\[ e^{-rt} \mathbb{E} \left[ 1_{\{\tau_{1,2} > t\}} (K - w_1 S_1^t + w_2 S_2^t)^+ \right] \]

\[ = e^{-rt} \sum_{n_1,n_2=1}^{\infty} e^{-\phi(\lambda_{n_1}^{1},\lambda_{n_2}^{2}) t} c_{n_1,n_2}(K) \varphi_{n_1}^{1}(x_1) \varphi_{n_2}^{2}(x_2) \]

- Where the expansion coefficient \( c_{n_1,n_2}(K) \) is given by,

\[ c_{n_1,n_2}(K) = \left( (K - w_1 x_1 - w_2 x_2)^+, \varphi_n(x) \right)_m \]

\[ = \int_{\mathbb{R}_+^2} (K - w_1 x_1 - w_2 x_2)^+ \varphi_{n_1}^{1}(x_1) \varphi_{n_2}^{2}(x_2) m_1(x_1) m_2(x_2) dx_1 dx_2 \]

\[ = K K \prod_{k=1}^{2} \left( \sqrt{\frac{\Gamma(n_k)}{\Gamma(n_k) |\mu_k + b_k|}} \frac{2 |\beta_k| A^{\nu_k+1}_k \tilde{K}^{2c_k-2\beta_k}_k}{\Gamma(\nu_k + 1)} \right) \]

\[ \times \sum_{p_1,p_2=0}^{\infty} (-1)^{p_1 + p_2} (\nu_1 + n_1)_p_1 (\nu_2 + n_2)_p_2 \frac{(A_1 \tilde{K}_{1}^{-2\beta_1})^{p_1} (A_2 \tilde{K}_{2}^{-2\beta_2})^{p_2}}{(\nu_1 + 1)_p_1! (\nu_2 + 1)_p_2!} \]

\[ \times \frac{\Gamma(2c_1 - 2\beta_1 (p_1 + 1)) \Gamma(2c_2 - 2\beta_2 (p_2 + 1))}{\Gamma(2c_1 - 2\beta_1 (p_1 + 1) + 2c_2 - 2\beta_2 (p_2 + 1) + 2)}. \]

where \( \tilde{K}_k = e^{-\rho_k t} K / \omega_k. \)
Two Firms Basket Put Option

- The single-name put on the stock $S^k$ that delivers the payoff if and only if the firm survives to maturity:

$$e^{-rt}E[1_{\{\tau_k > t\}}(K - w_k S^k_t)^+] = e^{-rt} \sum_{n=1}^{\infty} e^{-\phi_k(\lambda^k_n) t} p_n^k(K) \varphi_n^k(x_k)$$

1D Lévy Exp.
Two Firms Basket Put Option

- The single-name put on the stock $S^k$ that delivers the payoff if and only if the firm survives to maturity:

$$e^{-rt} \mathbb{E}\left[\mathbf{1}_{\{\tau_k > t\}} (K - w_k S^k_t)^+ \right] = e^{-rt} \sum_{n=1}^{\infty} e^{-\phi_k(\lambda_n^k) t} p_n^k(K) \varphi_n^k(x_k),$$

- Where the expansion coefficient $p_n^k(K)$ is given as,

$$p_n^k(K) = \left((K - w_k x_k)^+, \varphi_n^k(x_k)\right)_{m_k}$$

$$= \int_{\mathbb{R}_+} (K - w_k x_k)^+ \varphi_n^k(x_k) m_k(x_k) dx_k$$
Two Firms Basket Put Option

- The single-name put on the stock $S^k$ that delivers the payoff if and only if the firm survives to maturity:

$$e^{-rt}E\left[1_{\{\tau_k > t\}}(K - w_k S_t^k)^+\right] = e^{-rt} \sum_{n=1}^{\infty} e^{-\phi_k(\chi^k_n) t} p^k_n(K) \varphi^k_n(x_k),$$

- Where the expansion coefficient $p^k_n(K)$ is given as,

$$p^k_n(K) = \left((K - w_k x_k)^+, \varphi^k_n(x_k)\right)_{m_k}$$

$$= \int_{\mathbb{R}^+} (K - w_k x_k)^+ \varphi^k_n(x_k) m_k(x_k) dx_k$$

$$= K \sqrt{\frac{\Gamma(\nu_k + n)}{\Gamma(n) |\mu_k + b_k|}} \frac{A^2_{k+1} \tilde{K}^{2(c_k - \beta_k)}}{\Gamma(\nu_k + 1)} \times$$

$$\left\{ \frac{1}{(1 + c_k/|\beta_k|)} 2F_2 \begin{pmatrix} \nu_k + n, & \nu_k + n \\ \nu_k + 1, & \nu_k + 1 - \frac{1}{2|\beta_k|} \end{pmatrix} ; -A_k \tilde{K}_{-2\beta_k}^k \right\}$$

$$- \frac{1}{(\nu_k + 1)} 1F_1 \begin{pmatrix} \nu_k + n \\ \nu_k + 2 \end{pmatrix} ; -A_k \tilde{K}_{-2\beta_k}^k$$

where $1F_1$ and $2F_2$ are the Kummer confluent hypergeometric function and the generalized hypergeometric function, respectively; and $\tilde{K}_k = e^{-\rho_k t} K / w_k$. 

Rafael Mendoza
McCombs
Default Correlation
The single-name put on the stock $S^1$ that delivers the payoff if and only if both firms survive:

$$e^{-rt} \mathbb{E}\left[1_{\{\tau_{1,2} > t\}}(K - w_1 S^1_t)^+\right] = e^{-rt} \sum_{n_1, n_2 = 1}^{\infty} e^{-\phi(\lambda^1_{n_1}, \lambda^2_{n_2}) t} p^1_{n_1}(K) c^2_{n_2} \varphi^1_{n_1}(x_1) \varphi^2_{n_2}(x_2),$$

where $c^2_{n_2}$ are the coefficients of the expansion for the survival probability of the second stock and, $p^1_{n_1}(K)$ are the expansion coefficients for the single-name put on the first stock.
The single-name put on the stock $S^1$ that delivers the payoff if and only if both firms survive:

$$e^{-rt}E\left[1_{\{\tau_{1,2} > t\}}(K-w_1 S^1_t)^+ \right] = e^{-rt} \sum_{n_1,n_2=1}^{\infty} e^{-\phi(\lambda_{n_1}^1,\lambda_{n_2}^2) t} p_{n_1}^1(K) c_{n_2}^2 \varphi_{n_1}^1(x_1) \varphi_{n_2}^2(x_2),$$

where $c_{n}^2$ are the coefficients of the expansion for the survival probability of the second stock and,

$p_{n}^1(K)$ are the expansion coefficients for the single-name put on the first stock.