Pricing index-CDS options in a nonlinear filtering model

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Short recapitulation of the non-linear filtering model by Frey & Schmidt (2009) and some new additional results.

Give a very brief recapitulation of the index-CDS

Present practical formulas for the forward starting index-CDS spreads in the filtering model of Frey & Schmidt (2009).

Discuss calibration of the model using nonlinear-filter SDE and maximum-likelihood methods with market data on index-CDS spreads

Present some numerical results in our calibrated model

Give a short outline of options on index-CDS and how to price them in the model presented here.
We consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q})$ where $\mathbb{Q}$ is a risk-neutral measure. Below, all computations are under $\mathbb{Q}$.

The state of the economy is driven by an unobservable background factor process $X$ modelling the "true" state of the economy.

$X$ is modelled as finite-state Markov chain on state space $S^X = \{1, 2, \ldots, K\}$ with generator $Q$ and we define $\mathcal{F}^X_t = \sigma(X_s; s \leq t)$.

The states in $S^X = \{1, 2, \ldots, K\}$ are ordered so that state 1 represents the best state and $K$ represents the worst state of the economy.

Market participants only observe the "noisy" history of the state of the economy, i.e. $X_t$ with "noise".
The default times

- Consider $m$ obligors with default times $\tau_1, \tau_2, \ldots, \tau_m$

- Let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be the $\mathcal{F}_t^X$-default intensities for $\tau_1, \tau_2, \ldots, \tau_m$ where $\lambda_i : \{1, 2, \ldots, K\} \mapsto [0, \infty)$ for each obligor

- Hence, each default time $\tau_i$ is given by

\[
\tau_i = \inf \left\{ t > 0 : \int_0^t \lambda_i(X_s)ds \geq E_i \right\}.
\]

where $E_1, \ldots, E_m$ are iid, $E_i \sim \text{Exp}(1)$, and independent of $\mathcal{F}_\infty^X$.

- The default times $\tau_1, \tau_2, \ldots, \tau_m$ are conditionally independent given the information of the factor process $X$, that is $\mathcal{F}_\infty^X$.

- By definition of the state space, the mappings $\lambda_i(\cdot)$ are strictly increasing in $k \in \{1, 2, \ldots, K\}$, that is $\lambda_i(k) < \lambda_i(k + 1)$
The nonlinear filtering model, cont.

- Let \( Y_{t,i} = 1_{\{\tau_i \leq t\}} \) and \( Y_t = (Y_{t,1}, \ldots, Y_{t,m}) \) so that the pure portfolio default history is given by \( \mathcal{F}^Y_t = \sigma(Y_s; s \leq t) \)

- Market participants do not observe \( X_t \) directly, instead they observe \( Z_t \)

\[
Z_t = \int_0^t a(X_s)ds + B_t
\]

(2)

where \( B_t \) is a \( l \)-dimensional Brownian motion independent of \( X_t \) and \( Y_t \) and \( a(\cdot) \) is a function from \( \{1, 2, \ldots, K\} \) to \( \mathbb{R}^l \).

- We define \( \mathcal{F}^Z_t = \sigma(Z_s; s \leq t) \) and the information available for market participants denoted by ”the market filtration” \( \mathcal{F}^M_t \), is given by

\[
\mathcal{F}^M_t = \mathcal{F}^Y_t \vee \mathcal{F}^Z_t
\]

(3)

- So prices of securities are given as conditional expectation with respect to the market filtration \( \mathbb{F}^M = (\mathcal{F}^M_t)_{t \geq 0} \)
A central quantity is the filtering probabilities $\pi^k_t$ defined as
\[
\pi^k_t = \mathbb{Q} \left[ X_t = k \mid \mathcal{F}^M_t \right]
\] (4)
and we let $\pi_t \in \mathbb{R}^K$ be the row-vector $\pi_t = (\pi_1^t, \ldots, \pi^K_t)$.

The SDE describing the dynamics of $\pi^k_t$ is well known in nonlinear filtering theory (Kushner-Stratonovic) and connects to the innovation approach. Frey & Schmidt (2009) states the KS-SDE in a heterogeneous credit portfolio.

We only consider exchangeable credit portfolios, so that $\lambda_i(X_t) = \lambda(X_t)$ for each obligor and we let $\lambda \in \mathbb{R}^K$ be the row-vector $\lambda = (\lambda(1), \ldots, \lambda(K))$.

Let $N_t$ be the number of defaults up to time $t$ in the portfolio, that is
\[
N_t = \sum_{i=1}^{m} Y_{t,i} = \sum_{i=1}^{m} 1_{\{\tau_i \leq t\}}.
\]

The portfolio credit loss at $t$ is given by $L_t = \frac{(1-\phi)}{m} N_t$ where $\phi$ is the recovery rate for each obligor.
Kushner-Stratonovic equations in exchangeable portfolios

For \( j = 1, \cdots, l \), let \( \mu_{t,j} \) be a Brownian motion with respect to \( \mathcal{F}_{t}^{M} \). Then:

The Kushner-Stratonovic SDE in exchangeable credit portfolios

Consider a homogeneous credit portfolio with \( m \) obligors. Then, with notation as above, the processes \( \pi_{t}^{k} \) satisfies the following \( K \)-dimensional system of SDE-s,

\[
d\pi_{t}^{k} = \gamma^{k}(\pi_{t})dN_{t} + \pi_{t} \left( Qe_{k}^{\top} - \gamma^{k}(\pi_{t})\lambda^{\top}(m - N_{t}) \right) dt + \sum_{j=1}^{l} \alpha_{j}^{k}(\pi_{t})d\mu_{t,j} \tag{5}
\]

where \( \gamma^{k}(\pi_{t}) \) and \( \alpha^{k}(\pi_{t}) \) are given by

\[
\gamma^{k}(\pi_{t}) = \pi_{t}^{k} \left( \frac{\lambda(k)}{\pi_{t}^{\top} \lambda} - 1 \right) \quad \text{and} \quad \alpha^{k}(\pi_{t}) = \pi_{t}^{k} \left( a(k) - \sum_{n=1}^{K} \pi_{t}^{n}a(n) \right). \tag{6}
\]

and \( \alpha_{j}^{k}(\cdot) \) is the \( j \)-th component of \( \alpha^{k}(\cdot) \).

Here we set \( l = 1 \) and denote \( \mu_{t,1} \) by \( \mu_{t} \). We also let \( a(k) = c \cdot \ln \lambda(k) \).
Example of KS-SDE: \( K = 2, \ m = 125, \ l = 1 \)

A realization of the process \( X_t \)

A realization of the point process \( N_t \)

A trajectory of \( \pi_t^1 \) simulated with the Kushner–Strataonovich SDE
A very short recapitulation of the index-CDS

- An index-CDS on a portfolio of \( m \) obligors, entered at \( t \) with maturity \( T \), gives A protection on defaults among the \( m \) obligors from B up to time \( T \).

- A pays B a fixed fee \( S(t, T) \) multiplied what is left in the portfolio at each payment time which are done quarterly in the period \([t, T]\).

- \( S(t, T) \) is set so expected discounted cash-flows between A and B are equal at \( t \) and \( S(t, T) \) is called the index-CDS spread with maturity \( T - t \). Hence,

\[
S(t, T) = \frac{\mathbb{E} \left[ \int_t^T B(t, s) dL_s \mid \mathcal{F}_t^M \right]}{\frac{1}{4} \sum_{n=t_n}^{[4T]} B(t, t_n) (1 - \frac{1}{m} \mathbb{E} \left[ N_{t_n} \mid \mathcal{F}_t^M \right])}
\]

where \( B(t, s) = e^{-r(s-t)} \) for constant \( r \) and \( t_n = \frac{n}{4}, n_t = [4t] + 1 \).

- \( S(0, T) \) quoted on daily basis on the market for standardized credit portfolios where \( T = 3, 5, 7, 10 \), see e.g the iTraxx Series.
Given our nonlinear filtering model we can now state the following results.

Consider an index-CDS portfolio in the nonlinear filtering model. Then, with notation as above

\[
S(t, T) = (1 - \phi) \left( 1 - \pi_t \left( e^{Q_\lambda(T-t)} \left( I + r (Q_\lambda - rI)^{-1} \right) e^{-r(T-t)} - r (Q_\lambda - rI)^{-1} \right) 1 \right) \frac{1}{4} \sum_{n=n_t}^{[4T]} \pi_t e^{Q_\lambda(t_n - t)} 1 e^{-r(t_n - t)}
\]

(8)

where \( Q_\lambda = Q - I_\lambda \) and \( I_\lambda \) is a diagonal-matrix such that \((I_\lambda)_{k,k} = \lambda(k)\) and \( \pi_t = (\pi^1_t, \ldots, \pi^K_t) \).

Note that given \( \pi_t \) the formula for \( S(t, T) \) is compact and computationally tractable closed-form expressions in terms of \( \pi_t \) and \( Q_\lambda \).
Calibrating the models using index-CDS spread data

- **Task:** estimate $\theta = (Q, \lambda)$

- Let $\{S_M(t, T)\}_{t \in t^{(s)}}$ be a **historical time-series** of model spreads observed at $N^{(s)}$ sample time points $t^{(s)} = \{t_1^{(s)}, \ldots, t_N^{(s)}\}$ where $T = t + T_0$ for $t \in t^{(s)}$.

- For each $t \in t^{(s)}$ we set $S(t, T)(\omega) = S_M(t, T)$ and rewrite the pricing equation (8) as

$$
\pi_t(\omega)C_t(\theta, S_M(t, T))1 = 1 - \phi
$$

where $C_t(\theta, S_M(t, T))$ is known to us in terms of $\theta = (Q, \lambda)$ and $S_M(t, T)$

- By using $\pi_t(\omega)1 = 1$ with (9) we get **linear equation system** for $\pi_t(\omega)$, viz.

$$
A_t \pi_t^\top(\omega) = b
$$

So, for fixed $\theta = (Q, \lambda)$ and observed $S_M(t, T)$ and if $A_t^{-1}$ exists we can find $\pi_t(\omega) = (\pi_1^t(\omega), \pi_2^t(\omega), \ldots, \pi^K_t(\omega))$ by solving (10).

- We want to estimate $\theta = (Q, \lambda)$ with **maximum likelihood techniques** by using time-series data $\{S_M(t, T)\}_{t \in t^{(s)}}$, Eq. (10) and the KS-SDE (5)
Calibrating the model using index-CDS spread data, cont.

- Let us outline this approach when $K = 2$ (enough to study $\pi^1_t(\omega)$).

- Recall that $\pi^1_t(\omega)$ must satisfy
  
  \[ d\pi^1_t = \gamma^1(\pi_t) dN_t + \pi_t \left( Q e_1^T - \gamma^1(\pi_t) \lambda^T (m - N_t) \right) dt + \alpha^1(\pi_t) d\mu_t \]

  where $\mu_t$ is Brownian motion with respect to $\mathcal{F}^M_t$.

- We discretize (11) with $t_{n+1}^{(s)} - t_n^{(s)} = \Delta t$ and assume solution to the discrete SDE is same as solution to (11).

- Let $x_n = S_M(t_n^{(s)}, t_n^{(s)} + T_0)$. The discrete KS-SDE for a fixed $\omega \in \Omega$ is
  
  \[ \Delta \pi_{n,1}(\theta, x_n) = g(\theta, x_n) \Delta t + \alpha_1(\theta, x_n) \Delta \mu_{t_n^{(s)}} \]

  where $\Delta \pi_{n,1}(\theta, x_n)$, $\alpha_1(\theta, x_n)$ and $g(\theta, x_n)$ are known via $x_n$, $\theta = (Q, \lambda)$ (our sample contains no defaults, so $N_t(\omega) = 0$ for all $t \in t^{(s)}$).
The likelihood-function

- Note that R.H.S in (12) is conditionally normally distributed, viz.
  \[ g(\theta, x_n) \Delta t + \alpha_1(\theta, x_n) \Delta \mu_{t_n} \sim \mathcal{N}(g(\theta, x_n) \Delta t, (\alpha_1(\theta, x_n))^2 \Delta t). \]  
  (13)

- and \( \{g(\theta, x_n) \Delta t + \alpha_1(\theta, x_n) \Delta \mu_{t_n}\}_{n=1}^{N^{(s)}} \) are independent.

- Hence, the likelihood function \( L(\theta|x_1, \ldots, x_{N^{(s)}}) \) is
  \[
  L(\theta|x_1, \ldots, x_{N^{(s)}}) = \prod_{n=1}^{N^{(s)}} \frac{1}{\sqrt{2\pi(\alpha_1(\theta, x_n))^2 \Delta t}} \exp \left( -\frac{(\Delta \pi_{n,1}(\theta, x_n) - g(\theta, x_n) \Delta t)^2}{2(\alpha_1(\theta, x_n))^2 \Delta t} \right).
  \]

- By letting \( \ell(\theta|x_1, \ldots, x_{N^{(s)}}) = -\ln L(\theta|x_1, \ldots, x_{N^{(s)}}) \) we retrieve MLE-parameters \( \theta_{MLE} \) as \( \theta_{MLE} = \arg\min_\theta \ell(\theta|x_1, \ldots, x_{N^{(s)}}) \).

- Data: iTraxx Europe on-the-run series \( (T - t = 5\text{ years}), \text{ Nov 2007-Feb 2010}, \text{ with 596 observations, } \Delta t = 1/250, m = 125, r = 3\%, \phi = 40\% \).

- Result: \( \theta_{MLE} = (c, \lambda_1, \lambda_2, q_{12}, q_{21}) = (0.2939, 0.001, 0.09, 0.0098, 0.004) \)
Time-series $S_M(t, t + 5)$ and calibrated implied $\pi_t^1(\omega)$
Options on the index-CDS

- With the calibrated parameters $\theta_{\text{MLE}}$ we can price more complex instruments where the index-CDS is underlying, e.g. options on index CDS-s.

- An option on an index-CDS with inception date today, strike $K$ and exercise date $t$ with maturity $T$ gives $A$ the right to enter an index-CDS at time $t$ with spread $K$ and maturity $T - t$, sold by $B$.

- Moreover, $B$ also pays $A$ the accumulated credit loss $L_t$ at time $t$.

- The payoff $\Pi(t, T; K)$ for this option at time $t$ is
  \[ \Pi(t, T; K) = (PV(t, T) (S(t, T) - K) + L_t)^+ \]  \hspace{1cm} (14)

  where
  \[ PV(t, T) = \frac{1}{4} \sum_{n=n_t}^{\lceil 4T \rceil} B(t, t_n) \left( 1 - \frac{1}{m} \mathbb{E} \left[ N_{t_n} | F_t \right] \right). \]  \hspace{1cm} (15)

- It is not correct to use Black-Scholes when finding the price of the option.
Inserting relevant quantities from the filtering model into (14)-(15) yields

\[
\Pi(t, T; K) = \left(\pi_t \left[A(t, T) - KB(t, T)\right] 1 \left(1 - \frac{N_t}{m}\right) + \frac{(1 - \phi) N_t}{m}\right)^+ 
\]

where \(A(t, T)\) and \(B(t, T)\) are defined as

\[
A(t, T) = (1 - \phi) \left[I - e^{Q_\lambda(T-t)} \left(I + r (Q_\lambda - rI)^{-1}\right) e^{-r(T-t)} + r (Q_\lambda - rI)^{-1}\right] 
\]

and

\[
B(t, T) = \frac{1}{4} \sum_{n=n_t}^{[4T]} e^{Q_\lambda(t_n-t)} e^{-r(t_n-t)}. 
\]

**Valuation via MC-simulation:** We use e.g. \(\theta_{MLE}\) to simulate \(\pi_t\) and then (16) to find \(\Pi(t, T; K)\). Note that \(A(t, T)\) and \(B(t, T)\) are deterministic.
Options on the index-CDS: numerical example

t=9 months, T−t=5 years, S(0,5)=90 bp, π₀=83%, θ_{MLE} as in previous slides, \(10^3\) MC–simulations

**CDS–index call–option prices**
Thank you for your attention!