INFORMATION ASYMMETRY IN PRICING OF CREDIT DERIVATIVES.

Caroline HILLAIRET, CMAP, Ecole Polytechnique
Joint work with Ying JIAO, LPMA, Université Paris VII

6th World Congress of the Bachelier Finance Society, June 24, 2010.

This research is part of the Chair Financial Risks of the Risk Foundation, the Chair Derivatives of the Future sponsored by the Fédération Bancaire Française, and the Chair Finance and Sustainable Development sponsored by EDF and Calyon
OUTLINE OF THE TALK

1. INTRODUCTION

2. THE INFORMATIONAL STRUCTURE

3. PRICING UNDER THE HISTORICAL PROBABILITY

4. RISK NEUTRAL PRICING

5. NUMERICAL EXAMPLE
Introduction
The informational structure
Pricing under the historical probability
Risk neutral pricing
Numerical example
There exist two main approaches in the credit risk modelling: the structural approach and the reduced-form approach.

The two approaches are related by the accessibility of information on the underlying asset value process (Duffie-Lando 2001, Jeanblanc-Valchev 2005, Coculescu-Geman-Jeanblanc 2006, Guo-Jarrow-Zeng 2008), delayed or noisy observation of the underlying process,

on the default threshold (El Karoui 1999, Giesecke-Goldberg 2008), constant or random default barrier.

**AIM** study the impact of the information concerning the default threshold in the credit analysis, in addition to the partial observation of the underlying asset process.
\( (\Omega, \mathcal{A}, \mathbb{P}) \) probability space.

- \((X_t)_{t \geq 0}\) positive continuous-time process: asset value of the firm. 
- \(\mathbb{F} = (\mathcal{F}_t = \sigma(X_s, s \leq t))_{t \geq 0}\) satisfying the usual conditions
- Default threshold \(L\), random variable in \(\mathcal{A}\).
- Default time \(\tau\)

\[
\tau = \inf\{t : X_t \leq L\} \quad \text{where } X_0 > L
\]

with the convention that \(\inf \emptyset = +\infty\)

- We introduce the decreasing process \(X^*\) defined as

\[
X^*_t = \inf\{X_s, s \leq t\}.
\]

- We assume that the filtration \(\mathbb{F}\) is generated by a Brownian motion \(B\).
OUR FRAMEWORK

The manager of the firm has full information concerning the underlying asset of the firm and he chooses the default barrier.

⇒ $L$ is a random variable set by the manager of the firm.

The investors on the market have different levels of information on the fundamental process $(X_t)_{t \geq 0}$ and on the default threshold $L$.

- **Full information** knowledge of $(X_t)_{t \geq 0}$ and $L$ (manager of the firm)
- **Noisy full information** knowledge of $(X_t)_{t \geq 0}$ + noisy signal on $L$ + default time observable
- **Progressive information** knowledge of $(X_t)_{t \geq 0}$ + default time observable
- **Delayed information** delayed information on $(X_t)_{t \geq 0}$ + default time observable
Pricing Framework

Different level of information $\Rightarrow \begin{cases} \text{different filtration } (\mathcal{H}_t)_t \\ \text{different pricing measure } Q \end{cases}$

Evaluate a credit-sensitive derivative claim of maturity $T$:
the value process at time $t < \tau \wedge T$ is given by

\[
V_t = R_t \mathbb{E}_Q \left[ CR_T^{-1} 1_{\{\tau > T\}} + \int_t^T 1_{\{\tau > u\}} R_u^{-1} dG_u + Z_{\tau} 1_{\{\tau \leq T\}} R_\tau^{-1} \right| \mathcal{H}_t \right] \tag{1}
\]

where
- $C$ ($\mathcal{F}_T$-measurable) represents the payment at the maturity $T$ (if $\tau \geq T$)
- $G$ ($\mathbb{F}$-adapted) represents the dividend payment
- $Z$ ($\mathbb{F}$-predictable) represents the recovery payment at the default time $\tau$
- $R$ is the discount factor process.
Outline

1 Introduction

2 THE INFORMATIONAL STRUCTURE

3 Pricing under the historical probability

4 Risk neutral pricing

5 Numerical example
The manager knows the threshold $L$ ω-wise from the beginning. Thus his information is given as the initial enlargement of the filtration $\mathbb{F}$ with respect to $L$:

**ASSUMPTION** $(H^S)$

\[
\mathcal{G}^M = (\mathcal{G}_t^M)_{t \geq 0} \text{ with } \mathcal{G}_t^M := \mathcal{F}_t \vee \sigma(L).
\]

We assume that $\mathbb{P}(L \in \cdot | \mathcal{F}_t)(\omega) \sim \mathbb{P}(L \in \cdot)$ for all $t$ for $\mathbb{P}$ almost all $\omega \in \Omega$.

**EXAMPLES**

- Assumption $(H^S)$ is satisfied if $L$ is independent of $\mathcal{F}_\infty$.
- Another example in a finite time horizon $T < T'$:

\[
L = l_s 1_{[0,a]}(X_{T'}) + l_i 1_{[a,\infty]}(X_{T'}), \quad l_i < l_s
\]
RISK NEUTRAL PRICING MEASURE FOR THE FULL INFORMATION

PROPOSITION

There exists a $\mathcal{G}^M$ adapted process $(\rho^M_s(L))$ (called information drift) such that

- $Y^M(L) = \mathcal{E} \left( - \int_0^L \rho^M_s(L)(dB_s - \rho^M_s(L)ds) \right)$ is a probability density

- Any $(\mathcal{F}, \mathcal{Q})$-local martingale is an $(\mathcal{G}^M, Y^M(L)\mathcal{Q})$-local martingale.
The progressive information

This is the standard information structure in the credit risk analysis. Investors know at each time $t$ whether or not default has occurred. Thus their information is given as the progressive enlargement of filtration of $\mathbb{F}$ with respect to $\tau$:

**Assumption ($H^N$)**

$$\mathcal{G} = (\mathcal{G}_t)_{t \geq 0} \quad \text{with} \quad \mathcal{G}_t = \mathcal{F}_t \vee \sigma(\{\tau \leq s\}, s \leq t).$$

**Remark**

If $L$ is independent of $\mathcal{F}_\infty$, the standard ($H$)-hypothesis is satisfied: every $(\mathbb{F}, \mathbb{P})$ local martingale is also a $(\mathcal{G}, \mathbb{P})$ local martingale.
**THE DELAYED INFORMATION**

\[
\mathcal{F}_t^D := \begin{cases} 
\mathcal{F}_0 & \text{if } t \leq \delta(t), \\
\mathcal{F}_{t-\delta(t)} & \text{if } t > \delta(t), 
\end{cases}
\]

- constant delay time model: \( \delta(t) = \bar{\delta} \)
- discrete observation model: \( \delta(t) = t - t_i^{(m)}, t_i^{(m)} \leq t < t_{i+1}^{(m)} \) where 
  \[ 0 = t_0^{(m)} < t_1^{(m)} < \cdots < t_m^{(m)} = T \] are the only discrete dates on which the information is renewed.

The delayed information is the progressive enlargement of filtration of \( \mathbb{F}^D \) with respect to \( \tau \):

**ASSUMPTION (\( H^D \))**

\[
\mathcal{G}_t^D = (\mathcal{G}_t^D)_{t \geq 0} \text{ with } \mathcal{G}_t^D = \mathcal{F}_t^D \vee \sigma(\{\tau \leq s\}, s \leq t).
\]
The investor observes the value of the firm \((X_t)_t\) and receives a noisy signal \((L_s = f(L, \epsilon_s))_{s \geq 0}\) on the threshold \(L\).

**Assumption \((H^N)\)**

\[
\mathcal{G}_t^I = (\mathcal{F}_t^I \vee \sigma(\{\tau \leq s\}, s \leq t))_{t \geq 0} \quad \text{with} \quad \mathcal{F}_t^I = \bigcap_{u \geq t} (\mathcal{F}_u \vee \sigma(f(L, \epsilon_s), s \leq u))
\]

- \(f : \mathbb{R}^2 \to \mathbb{R}\) is a given measurable function.
- \(\epsilon = \{\epsilon_t, t \geq 0\}\) is independent of \(\mathcal{F}_\infty \vee \sigma(L)\).
- \(\mathbb{P}(L \in \cdot | \mathcal{F}_t)(\omega) \sim \mathbb{P}(L \in \cdot)\) for all \(t\) for \(\mathbb{P}\) almost all \(\omega \in \Omega\).

**Example**

Example in a finite time horizon \(T < T'\):

\(L_s = L + W_{g(T'-s)}\) with \(W\) an independent Brownian motion and

\(g : [0, T'] \to [0, \infty[\) a strictly increasing bounded function with \(g(0) = 0\).
PROPOSITION

There exists a $\mathbb{F}^I$ adapted process $(\rho^I)$ such that

- $Y^I = \mathcal{E}(\int_0^\cdot \rho^I_s (dB_s - \rho^I_s ds))$ is a probability density
- Any $(\mathbb{F}, \mathbb{Q})$-local martingale is an $(\mathbb{F}^I, Y^I \mathbb{Q})$-local martingale.
Outline

1. Introduction
2. The informational structure
3. Pricing under the historical probability
4. Risk neutral pricing
5. Numerical example

Information asymmetry in Pricing of Credit Derivatives
**AIM** Compute the price of the contingent claim \((C, G, Z)\) with maturity \(T\) given different sources of information:

\[
V_t = R_t \mathbb{E}_\mathbb{P} \left[ CR_T^{-1} 1_{\{\tau > T\}} + \int_{t}^{T} 1_{\{\tau > u\}} R_u^{-1} dG_u + Z_{\tau} 1_{\{\tau \leq T\}} R_\tau^{-1} \bigg| \mathcal{H}_t \right]
\]

\[\rightarrow \mathbb{P} \text{ is the historical probability measure}\]

\[\rightarrow (\mathcal{H}_t)_{t \geq 0} \text{ describes the accessible information for the investors.}\]

- \((\mathcal{H}_t)_{t \geq 0} = (\mathcal{G}_t^M)_{t \geq 0}\) for the full information,
- \((\mathcal{H}_t)_{t \geq 0} = (\mathcal{G}_t^I)_{t \geq 0}\) for a noisy full information.
- \((\mathcal{H}_t)_{t \geq 0} = (\mathcal{G}_t)_{t \geq 0}\) for the progressive information.
- \((\mathcal{H}_t)_{t \geq 0} = (\mathcal{G}_t^D)_{t \geq 0}\) for the delayed information.
Pricing for the Full Information

Proposition

We define \( F_t^M(x) := p_t(x)1_{\{x^*>x\}} \) where \( p_t(x)(\omega) = \frac{dP_t^L(\omega, x)}{dP_L(\omega, x)} \), \( P_t^L(\omega, dx) \) = a regular version of the conditional law of \( L \) given \( \mathcal{F}_t \), \( P_L^L = \) the law of \( L \).

The value process of the contingent claim \((C, G, Z)\) given the full information \( (\mathcal{G}_t^M)_{t \geq 0} \) is

\[
V_t^M = 1_{\{\tau>t\}} \frac{\tilde{V}_t^M(L)}{p_t(L)}
\]

where

\[
\tilde{V}_t^M(L) = R_t E_P \left[ CR_T^{-1} F_T^M(x) + \int_t^T F_s^M(x) R_s^{-1} dG_s - \int_t^T Z_s R_s^{-1} dF_s^M(x) \bigg| \mathcal{F}_t \right]_{x=L}.
\]
Pricing for the Progressive and Delayed Information

**Proposition**

We define $S_t := \mathbb{P}(\tau > t \mid \mathcal{F}_t)$.

- The value process given the progressive information flow $\mathbb{G}$ is
  \[
  V_t = 1_{\{\tau > t\}} \frac{R_t}{S_t} \mathbb{E}_\mathbb{P}\left[ R_T^{-1} S_T C + \int_t^T R_u^{-1} S_u dG_u - \int_t^T R_u^{-1} Z_u dS_u \mid \mathcal{F}_t \right].
  \]

- The value process for a delay-informed investor is
  \[
  V_t^D = \frac{1_{\{\tau > t\}}}{\mathbb{E}[S_t \mid \mathcal{F}_t^D]} \mathbb{E}_\mathbb{P}\left[ \frac{R_t}{R_T} S_T C + \int_t^T \frac{R_t}{R_u} S_u dG_u - \int_t^T \frac{R_t}{R_u} Z_u dS_u \mid \mathcal{F}_t^D \right].
  \]
Pricing for the Noisy Information

**Proposition**

We assume \((H^N)\) with \(L_t = L + \epsilon_t\), \(\epsilon_t\) being a continuous process with backwardly independent increments and whose marginal has density \(q_t\). The value process for the noisy full information flow \(\mathcal{G}^I\) is given by

\[
V^I_t = \frac{1_{\{\tau > t\}}}{\int_{\mathbb{R}} F^M_t(l) q_t(L_t - l) P^L(dl)} \int \tilde{V}^M_t(l) q_t(L_t - l) P^L(dl)
\]

where \(\tilde{V}^M\) and \(F^M\) are defined for the full information.

\[
F^M_t(x) := p_t(x) 1_{\{X^*_t > x\}}.
\]

\[
\tilde{V}^M_t(L) = R_t E_{\mathbb{P}}\left[ CR_T^{-1} F^M_T(x) + \int_t^T F^M_s(x) R_s^{-1} dG_s - \int_t^T Z_s R_s^{-1} dF^M_s(x) \bigg| \mathcal{F}_t \right]_{x=L}.
\]
OUTLINE

1 Introduction
2 The informational structure
3 Pricing under the historical probability
4 Risk neutral pricing
5 Numerical example
Risk neutral probabilities

- We assume that a pricing probability $\mathbb{Q}$ is given with respect to the filtration $\mathbb{F}$ of the fundamental process $X$ (for example, $\mathbb{Q}$ such that $X$ is an $(\mathbb{F}, \mathbb{Q})$ local martingale).
- We want to focus on the change of probability measures due to the different sources of informations and on its impact on the pricing of credit derivatives, ⇒ without loss of generality, we take the historical probability $\mathbb{P}$ to be the benchmark pricing probability $\mathbb{Q}$ on $\mathbb{F}$ and $\mathbb{G}$.
- The pricing probability for the manager is $\mathbb{Q}^M$ where $\frac{d\mathbb{Q}^M}{d\mathbb{Q}} = Y^M(L)$ with $Y^M(L) = \mathcal{E}(-\int_0^L \rho_s^M(L)(dB_s - \rho_s^M(L)ds))$.
- The pricing probability for the noisy full information is $\mathbb{Q}^I$ where $\frac{d\mathbb{Q}^I}{d\mathbb{Q}} = Y^I$ with $Y^I = \mathcal{E}(-\int_0^I \rho_s^I(dB_s - \rho_s^I ds))$.
- We also take $\mathbb{Q}$ as the pricing probability for the delayed information.
Proposition

1) Define $F_t^{QM}(l) = 1\{x_t^* > l\}$. Then the value process of a credit sensitive claim $(C, G, Z)$ for the manager’s full information under the risk neutral probability measure $Q^M$ is given by

$$V_t^{QM} = 1\{\tau > t\}R_t \mathbb{E}_P\left[CR_T^{-1}F_T^{QM}(x)\right]$$

$$+ \int_t^T F_s^{QM}(x)R_s^{-1}dG_s - \int_t^T Z_sR_s^{-1}dF_s^{QM}(x) \mid \mathcal{F}_t \big|_{x=L}.$$
2) Let ε be a continuous process with backwardly independent increments such that the probability law of εₜ has a density qₜ(·) w.r.t. the Lebesgue measure. Let µₜ,θ be the probability law of εθ − εₜ. Then the value process for the insider’s noisy full information under Q_I is given by

\[ V^Q_I(t) = \frac{1_{\{\tau > t\}}}{\int_{\mathbb{R}} F^M_t(l) q_t(L_t - l)P^L(dl)} \int \tilde{V}^Q_I(l) q_t(L_t - l)P^L(dl) \]

where

\[ \tilde{V}^Q_I(l) = R_t \mathbb{E}_P \left[ CR_T^{-1} F^I_{t,T}(u, l) + \int_t^T F^I_{t,\theta}(u, l) R_\theta^{-1} dG_\theta \right. \]

\[ \left. - \int_t^T R_\theta^{-1} Z_\theta dF^I_{t,\theta}(u, l) | \mathcal{F}_t \right]_{u=L_t}, \]

\[ F^I_{t,\theta}(u, l) = \mathcal{E} \left( \int_t^\theta \int \rho^I_s(u + y) \mu_{t,\theta}(dy) dB_s \right)^{-1} F^M_t(l). \]
Introduction

The informational structure

Pricing under the historical probability

Risk neutral pricing

Numerical example

Outline

1. Introduction

2. The informational structure

3. Pricing under the historical probability

4. Risk neutral pricing

5. Numerical example
Let $L$ be a $(\mathcal{F}_t)_{t \geq 0}$-independent random variable taking two values $l_i, l_s \in \mathbb{R}$, $l_i \leq l_s$ such that

$$\mathbb{P}(L = l_i) = \alpha, \quad \mathbb{P}(L = l_s) = 1 - \alpha \quad (0 < \alpha < 1).$$

We suppose that the asset values process $X$ satisfies the Black Scholes model:

$$\frac{dX_t}{X_t} = \mu dt + \sigma dB_t, \quad t \geq 0.$$

We compute the value process of a defaultable bond.
Value process of a defaultable bond

Numerical values in the following simulations: \( l_i = 1, l_s = 3, \alpha = \frac{1}{2} \).

![Graphs showing dynamic price of the defaultable bond and firm value](image)

**Figure:** \( L = l_i \)
Introduction

The informational structure

Pricing under the historical probability

Risk neutral pricing

Numerical example

**Figure:** $L = l_s$

dynamic price of the defaultable bond  

firm value
We compare the dynamic price of the defaultable bond, for the same scenario of the firm value but depending on the level of the threshold fixed by the manager.

\[ L = l_s \text{ or } L = l_i \]
REFERENCES