DERIVATIVE TIME SCALE
PERTURBATIONS

Knut Sølna, UC Irvine

Collaborators:
Jean-Pierre Fouque, UC Santa Barbara
Ronnie Sircar, Princeton
George Papanicolaou, Stanford
OUTLINE AND OBJECTIVES

¬ Stochastic volatility modeling.
¬ Illustration equities.
¬ On model and parameters interpretation.
¬ Role of skew in credit markets.

Aspects and objectives:

◊ “Hidden” Volatility/Parameter Time Scales and parameter heterogeneity are important; leverage and clustering effects.
◊ Efficient and simple description of Stochastic Volatility effects using Perturbation Methods, under separation of time scales.
◊ Parsimonious “Effective Parameteric” representation for derivative Linkage and insight captured by perturbations.
Volatility Time Scales

• Rescale the time of a diffusion process \( Y_t^1 \):

\[
Y_t^\alpha = Y_{\alpha t}^1
\]

\( \alpha \) large \rightarrow “speeding up” the process \( Y_t^1 \)
\( \alpha \) small \rightarrow “slowing down” the process \( Y_t^1 \)

\( 1/\alpha \) is the characteristic time scale of the process \( Y_t^\alpha \).

• On effective volatility, fast scale case:

\[
\overline{\sigma^2}(0, T) \equiv \frac{1}{T} \int_0^T f^2(Y_t^\alpha) dt = \frac{1}{T'} \int_0^{T'} f^2(Y_s^1) ds, \quad T' \equiv \alpha T \rightarrow +\infty
\]

Assuming that \( Y^1 \) is ergodic with invariant distribution \( \Phi_Y \) then:

\[
\lim_{T' \rightarrow +\infty} \frac{1}{T'} \int_0^{T'} f^2(Y_s^1) ds = \int f^2(y) \Phi(dy) \equiv \langle f^2 \rangle_{\Phi_Y}.
\]

Effective volatility: \( \overline{\sigma^2} \equiv \langle f^2 \rangle_{\Phi_Y} \).
\[
dY_t^1 = c(Y_t^1)dt + g(Y_t^1)dW_t, \quad Y_0^1 = y
\]

\[
dY_t^\alpha \overset{\mathcal{D}}{=} \alpha c(Y_t^\alpha)dt + \sqrt{\alpha} g(Y_t^\alpha)dW_t, \quad Y_0^\alpha = y
\]

Fast “oscillating” integral:

\[
\mathcal{H}(\alpha; T) \equiv \frac{1}{T} \int_0^T \left( f^2(Y_s^\alpha) - \bar{\sigma}^2 \right) ds = \frac{1}{T} \int_0^T \mathcal{L}_{Y_1^1} \phi(Y_s^\alpha) ds,
\]

Poisson equation: \( \mathcal{L}_{Y_1^1} \phi(y) = f^2(y) - \bar{\sigma}^2 \)

\[
\text{Itô:} \quad d\phi(Y_s^\alpha) = \alpha \mathcal{L}_{Y_1^1} \phi(Y_s^\alpha) ds + \sqrt{\alpha} \phi'(Y_s^\alpha) g(Y_s^\alpha) dW_s
\]

\[
\mathcal{H}(\alpha; T) = \frac{1}{\alpha T} \int_0^T d\phi(Y_s^\alpha) - \frac{1}{\sqrt{\alpha} T} \int_0^T \phi'(Y_s^\alpha) g(Y_s^\alpha) dW_s
\]

\[
= -\frac{1}{\sqrt{\alpha}} \left( \frac{1}{T} \int_0^T \phi'(Y_s^\alpha) g(Y_s^\alpha) dW_s \right) + O \left( \frac{1}{\alpha} \right)
\]

Correction:

(i) Correlation with the BM driving the underlying gives correction at the order \( \sqrt{\alpha} \). (ii) Risk neutral Market price of volatility risk gives correction at same order.
Assuming $f$ and $g$ smooth,

$$\sigma^2(0, T) = \frac{1}{T} \int_0^T f^2(Y_t^\alpha) dt \longrightarrow f^2(y) \quad \text{as} \quad \alpha \to 0$$

$$\tilde{H}(\alpha; T, y) = \frac{1}{T} \int_0^T \left( f^2(Y_s^\alpha) - f^2(y) \right) ds$$

$$= \frac{1}{T} \int_0^T \left[ f^2 \left( y + \alpha \int_0^t c(Y_s^\alpha) ds + \sqrt{\alpha} \int_0^t g(Y_s^\alpha) dW_s \right) - f^2(y) \right] dt$$

$$= 2\sqrt{\alpha} f(y)f'(y)g(y) \left( \frac{1}{T} \int_0^T W_t dt \right) + O(\alpha)$$

**Correction**: (i) Correlation with the BM driving the underlying gives correction at the order $\sqrt{\alpha}$ (ii) Risk neutral Market price of volatility risk gives correction at same order.
The volatility is a function of two factors:

\[ \sigma_t = f(Y_t, Z_t) \]

- \( Y_t \) fast mean-reverting (ergodic):

\[
dY_t = \frac{1}{\varepsilon}\alpha(Y_t)dt + \frac{1}{\sqrt{\varepsilon}}\beta(Y_t)dW_t^{(y)}, \quad 0 < \varepsilon \ll 1
\]

- \( Z_t \) is slowly varying:

\[
dZ_t = \delta c(Z_t)dt + \sqrt{\delta}g(Z_t)dW_t^{(z)}, \quad 0 < \delta \ll 1
\]

\((y, z)\) will denote the initial point for \((Y, Z)\)

**Local Effective Volatility:**

\[
\bar{\sigma}^2(z) \equiv \left\langle f^2(\cdot, z) \right\rangle_{\Phi_Y}
\]

Fouque et al, SIAM MMS 03.
Consider (zero short rate):

\[ dX_t = f(Y_t)X_t dW_t^{(x)}; \quad dY_t = \frac{1}{\varepsilon} \alpha(Y_t) dt + \frac{1}{\sqrt{\varepsilon}} \beta(Y_t) dW_t^{(y)}, \]

\[ d \langle W^{(x)}, W^{(y)} \rangle = \rho_y. \]

- **The price**: \( P_t = \mathbb{E}[h(X_T) \mid \mathcal{F}_t]. \)

- **The approximation**: \( \tilde{P}_t = M_t + R_t \) with
  (i) \( \tilde{P}_t = \tilde{P}(t, X_t), \tilde{P}_T = h(x), \) (ii) \( M_t \) martingale, (iii) \( R_t = O(\varepsilon). \)

Then:

\[ \tilde{P}_t = \mathbb{E}^*[M_T + R_T \mid \mathcal{F}_t] + R_t - \mathbb{E}^*[R_T \mid \mathcal{F}_t] \]
\[ = P_t + (R_t - \mathbb{E}^*[R_T \mid \mathcal{F}_t]) = P_t + O(\varepsilon). \]
Fast Scale Approximation

\[ \tilde{P}_t - \tilde{P}_0 = \int_0^t \mathcal{L}_{BS}(\bar{\sigma}) \tilde{P}_s \, ds + f(Y_s) X_s \partial_x \tilde{P}_s dW_s^{(x)} \]

by Poisson

\[ \equiv \int_0^t \mathcal{L}_{BS}(\bar{\sigma}) \tilde{P}_s \, ds + f(Y_s) X_s \partial_x \tilde{P}_s dW_s^{(x)} \]

\[ + \frac{\varepsilon}{2} \left( d\phi - \frac{\beta(Y_s) \phi'(Y_s)}{\sqrt{\varepsilon}} \right) dW(y) X_s^2 \partial_x^2 \tilde{P}_s \]

by parts

\[ \equiv \int_0^t \mathcal{L}_{BS}(\bar{\sigma}) \tilde{P}_s \, ds - \sqrt{\varepsilon} \left\langle \mathcal{V}(Y_s) \right\rangle_{\Phi_Y} d s (X_t \partial_x) (X_t^2 \partial_x^2) \tilde{P}_t \]

\[ + \sqrt{\varepsilon} \left( \mathcal{V}(Y_s) - \left\langle \mathcal{V}(Y_s) \right\rangle_{\Phi_Y} \right) (X_s \partial_x) (X_s^2 \partial_x^2) \tilde{P}_s \, ds + \ldots dW_s^{(x)} + \ldots dW_s^{(y)} \]

\[ = \text{martingale} + O(\varepsilon). \]

\[ d \left\langle \phi, x^2 \partial_x^2 P \right\rangle_t = \rho_y \beta(Y_t) f(Y_t) \phi'(Y_s) (X_t \partial_x)(X_t^2 \partial_x^2) P_t \, dt \]

\[ \equiv \mathcal{V}(Y_t)(X_t \partial_x)(X_t^2 \partial_x^2) P_t \, dt. \]
Thus we want \( \bar{P} = \bar{P}(t, x) \) so that:

\[
L_{BS}(\bar{\sigma}) \bar{P}_t - \sqrt{\varepsilon} \langle \mathcal{V}(Y_t) \rangle_{\Phi_Y} (x \partial_x) (x^2 \partial_x^2) \bar{P}_t = O(\varepsilon); \bar{P}_T = h.
\]

- The case with market price of risk:

\[
dY_t = \left( \frac{1}{\varepsilon} \alpha(Y_t) - \frac{1}{\sqrt{\varepsilon}} \beta(Y_t) \Lambda(Y_t) \right) dt + \frac{1}{\sqrt{\varepsilon}} \beta(Y_t) dW_t^{(y)^*},
\]

which gives \( \varepsilon d\phi \mapsto \varepsilon d\phi + \sqrt{\varepsilon} \beta(Y_t) \Lambda(Y_t) \Phi'(Y_t) dt. \)

Then we want

\[
L_{BS}(\bar{\sigma}) \bar{P}_t - \frac{\sqrt{\varepsilon}}{2} \left\langle \beta(Y_t) f(Y_t) \Phi'(Y_t) \right\rangle_{\Phi_Y} (x \partial_x) (x^2 \partial_x^2) \bar{P}_t
\]

\[
+ \frac{\sqrt{\varepsilon}}{2} \left\langle \beta(Y_t) \Lambda(Y_t) \Phi'(Y_t) \right\rangle_Y (x^2 \partial_x^2) P_t = O(\varepsilon); \bar{P}_T = h.
\]
• Recall therefore:

\[ \mathcal{L}_{BS}(\bar{\sigma})\tilde{P}_t + \sqrt{\varepsilon} \left( V_3(x\partial_x)(x^2\partial_x^2)\tilde{P}_t + V_2(x^2\partial_x^2)P_t \right) = O(\varepsilon), \]
\[ \tilde{P}_T = h. \]

\[ \rightarrow \textbf{Decomposition} \quad \tilde{P} = P^{(0)} + \sqrt{\varepsilon}P^{(1)} + O(\varepsilon). \]

\[ \mathcal{L}_{BS}(\bar{\sigma})(P^{(0)}) = 0; \quad P^{(0)}|_T = h, \]
\[ \mathcal{L}_{BS}(\bar{\sigma})(P^{(1)}) = (\sqrt{\varepsilon}A)P^{(0)}; \quad P^{(1)}|_T = 0. \]

\[ A = -V_2(x^2\partial_x^2) - V_3(x\partial_x)(x^2\partial_x^2), \]
\[ V_2 = \frac{1}{2} \langle \beta \Lambda \phi' \rangle_{\Phi_Y}, \quad V_3 = -\frac{\rho_y}{2} \langle \beta f \phi' \rangle_{\Phi_Y}. \]

\[ \rightarrow \textbf{Can solve explicitly for } \tilde{P}^{(0)} \text{ and } \tilde{P}^{(1)}! \]
Now $\sigma = f(Z_t)$ with $\delta \ll 1$:

$$dZ_t = \left( \delta c(Z_t) - \sqrt{\delta} g(Z_t) \Lambda(Z_t) \right) dt + \sqrt{\delta} g(Z_t) dW_t^{(z)}$$

$$d\langle W(x), W(z) \rangle = \rho_z .$$

Slow scale does not go away via averaging (rather “freezes”): $\tilde{P} = \tilde{P}(t, x, z)$, now we need:

$$\tilde{P}_t - \tilde{P}_0 = \int_0^t \mathcal{L}_{BS}(f(Z_s))\tilde{P}_s ds + \left( \sqrt{\delta} \mathcal{M}_1(Z_s) + \delta \mathcal{M}_2(Z_s) \right) \tilde{P}_s$$

$$+ f(Z_s)X_s \partial_x \tilde{P}_s dW_s^{(x)} + \sqrt{\delta} g(Z_t) \partial_z \tilde{P}_s dW_s^{(z)} = \text{martingale} + O(\varepsilon) ,$$

with

$$\mathcal{M}_1 = g(z) \left( \rho_z f(z)x \partial_{xz}^2 - \Lambda(z) \partial_z \right) , \quad \mathcal{M}_2 = \frac{1}{2} g^2(z) \partial_z^2 + c(z) \partial_z .$$

$$\Rightarrow \mathcal{L}_{BS}(\sigma(z)) \tilde{P}_t + \sqrt{\delta} \mathcal{M}_1(z) \tilde{P}_t = 0 , \quad \tilde{P}_T = h .$$

Can solve explicitly for $\tilde{P}$ again.
\[ \sigma = \sigma(Y_t, Z_t). \]

- **Decomposition** \( \tilde{P} = P^{(0)} + P^{(y)} + P^{(z)} + \mathcal{O}(\varepsilon). \)

\[
\begin{align*}
\mathcal{L}_{BS}(\bar{\sigma}(z))(P^{(0)}) &= 0; & P^{(0)}|_T &= h, \\
\mathcal{L}_{BS}(\bar{\sigma}(z))(P^{(y)}) &= \mathcal{A}^{\varepsilon}P^{(0)}; & P^{(y)}|_T &= 0, \\
\mathcal{L}_{BS}(\bar{\sigma}(z))(P^{(z)}) &= -\langle \mathcal{M}_1^\delta \rangle_{\Phi_Y} P^{(0)}; & P^{(z)}|_T &= 0,
\end{align*}
\]

\[
\mathcal{A} = -V_2^\varepsilon(x^2 \partial_x^2) - V_3^\varepsilon(x \partial_x)(x^2 \partial_x^2),
\]

\[
V_2^\varepsilon(z) = \frac{\sqrt{\varepsilon}}{2} \langle \beta \Lambda_y \phi' \rangle_{\Phi_Y},
V_3^\varepsilon(z) = -\frac{\sqrt{\varepsilon \rho_y}}{2} \langle \beta f \phi' \rangle_{\Phi_Y},
\]

\[
\langle \mathcal{M}_1^\delta \rangle_{\Phi_Y} = \sqrt{\delta} \rho_z g \langle f \rangle_{\Phi_Y} x \partial_{xz}^2 - \sqrt{\delta} g \langle \Lambda_z \rangle_{\Phi_Y} \partial_z,
\]

\[
= V_1^\delta x \partial_{xz}^2 + V_0^\delta \partial_z.
\]

In practice: (i) \( \partial_z \mapsto \bar{\sigma}' \partial_\sigma. \) (ii) Eliminate \( V_2 \) via parameter reduction step.
Consider the situation of a call:
The implied volatility $I$:

$$I \approx (b^* + \tau b^\delta) + \left(a^\varepsilon + \tau a^\delta\right)\frac{\log K/X}{\tau},$$

coefficients $a^\varepsilon, b^*, a^\delta, b^\delta$ affine functions of the $V$'s.

calibrate $V$'s from implied volatility.

The implied volatility surface approximation for $(a^\varepsilon, b^*, a^\delta, b^\delta) = (-.2, .2, -.015, -.08)$.

For a fixed time to maturity the surface is affine in $\log(K/x)$. 
Long dated options and multiscale dynamics

\[ dX_t = \mu X_t dt + f(Y_t, Z_t) X_t dW_t^{(x)} \]
\[ dY_t = \frac{1}{\varepsilon} \alpha(Y_t) dt + \sqrt{\frac{1}{\varepsilon}} \beta(Y_t) dW_t^{(y)} \]
\[ dZ_t = \delta c(Z_t) dt + \sqrt{\delta} g(Z_t) dW_t^{(z)} \]

with \( d\langle W^{(x)}, W^{(y)} \rangle_t = \rho_y dt \), \( d\langle W^{(x)}, W^{(z)} \rangle_t = \rho_z dt \) and \( \varepsilon \ll T \ll \frac{1}{\delta} \).

\textbf{Right} : calibration wrt \( a^{\varepsilon}, b^* \); \textbf{Left} : calibration wrt \( a^{\varepsilon}, b^*, a^\delta, b^\delta \).

Fouque et al, SIAM MMS 03.
• An average strike option:

\[ h_a = \left( X_T - \frac{1}{T} \int_0^T X_s ds \right)^+, \]
\[ \leftrightarrow dI_t = X_t dt, \quad \mathcal{L}_2 \mapsto \hat{\mathcal{L}}_2 = \mathcal{L}_2 + x \partial I. \]

• The price correction solves

\[ \langle \hat{\mathcal{L}}_2 \rangle \hat{P}_1 = \mathcal{L}_s \hat{P}_0, \]
\[ \hat{P}_1(T, x, v, I) = 0, \]

where

\[ \mathcal{L}_s := \left\{ \mathcal{A}^\varepsilon - \langle \mathcal{M}^\delta \rangle_{\Phi_Y} \right\}, \]

and \( \mathcal{L}_s \) is determined by the calibrated effective market parameters \( a^\varepsilon, b^*, a^\delta, b^\delta \).
Approximation and Approach Stability

Example Periodicity: Volatility as function of trading hour:

\[ dX_t = \mu X_t dt + f(Y_t, t/\varepsilon) X_t dW_t^{(x)}. \]

- Averaging functional \( \langle \cdot \rangle_{\phi_Y} \) modified to include averaging over periodic cycle.
- The structure of the approximation is not changed only the parameter interpretation is.

Jump component in fast scale; again modified averaging with respect invariant distribution of the jump process.

Non Markovian models; use “conditional shift”

\[ \phi(Y_t) \mapsto -\frac{1}{\varepsilon} \mathbb{E} \left[ \int_t^T (f^2(Y_s) - \bar{\sigma}^2) \, ds \mid \mathcal{F}_t \right]. \]
Some References

Derivatives in financial markets with stochastic volatility; Fouque, Papanicolaou & Sircar; Cambridge 2000.

Singular perturbation in option pricing; Fouque, Papanicolaou, Sircar & Solna; SIAP 2003.

Multiscale stochastic volatility asymptotics; Fouque, Papanicolaou, Sircar & Solna; SIAM MMS 2003.

On the short-time behavior of the implied volatility for jump-diffusion models with stochastic volatility; Alos, Leon & Vives; Finance Stoch 2007.

Asymptotic Analysis for Stochastic Volatility: Martingale Expansion; Fukasawa; preprint 2010.
• **Model under risk neutral**:

\[
\begin{align*}
\frac{dX_t}{dt} &= rX_t dt + f(Y_t)X_t dW_t^{(x)*} \\
\frac{dY_t}{dt} &= \left(\frac{1}{\varepsilon}\alpha(Y_t) - \frac{1}{\sqrt{\varepsilon}}\beta(Y_t)\Lambda(Y_t)\right) dt + \beta(Y_t) dW_t^{(y)*},
\end{align*}
\]

where the Brownian motions \( W^{(*)*} \) have covariation \( \rho_y \) and \( \Lambda \) is a combined market price of risk parameter.

• **Absorption**:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \tilde{\mu} \frac{\partial u}{\partial s} + \tilde{\sigma}^2 \frac{\partial^2 u}{\partial s^2} + \text{sign}(\rho) v \frac{\partial^3 u}{\partial s^3}, \\
u(0, s) &= e^{-rT} h(e^s) := \tilde{h}(s), \\
\tilde{\mu} &= r - \frac{\bar{\sigma}^2}{2} + V_2^\varepsilon \\
\tilde{\sigma}^2 &= \frac{\bar{\sigma}^2}{2} + (V_2^\varepsilon - V_3^\varepsilon) > 0 \\
v &= |V_3^\varepsilon| \neq 0.
\end{align*}
\]
• **Krylov** considered:

\[
\frac{\partial u}{\partial t} = (-1)^{q+1} \frac{\partial^2 q u}{\partial s^2 q}.
\]

• Wiener measure for \( q = 1 \). • In general define a signed measure on path space. • Deduced Feynman-Kac and used this for deducing a type of “arc-sine law”.

• **Orsingher** considered:

\[
\frac{\partial u}{\partial t} = -\frac{\partial^3 u}{\partial s^3},
\]

whose fundamental solution gives a non-symmetric signed measure and can similarly be associated with “Orsinger canonical” pseudo processes. • Faynman-Kac still valid. • Deduced an “asymmetric version” of the arc-sine law, time epochs of negative path values partly cancels.
Pseudo-Process Representation

Let \((B_t)_{0 \leq t \leq T}\) be a Brownian motion and \((X_t)_{0 \leq t \leq T}\) a pseudo-process defined on a signed probability space \((\Omega^*, \mathcal{F}^*, P^*)\) (sum of Orsingher canonical process and Brownian motion on product space) with density:

\[
p(t, x) = \frac{1}{\sqrt[3]{3vt}} \mathbb{E} \left[ Ai \left( \frac{-(x + \tilde{\mu}t) \text{sign}(\rho) - \sqrt{2\tilde{\sigma}^2 B_t}}{\sqrt[3]{3vt}} \right) \right],
\]

for all \(0 < t \leq T\) and \(p(0, .) = \delta_0\), such that for all \(0 \leq t \leq T\),

\[
u(t, x) = \mathbb{E}^* \left[ \tilde{h}(x + X_t) \right],
\]

where \(\mathbb{E}^*\) denotes the expectation (the integral) with respect to the signed measure and \(Ai\) is the Airy function.

For fixed \((t, x) \in [0, T] \times [0, +\infty[\), the price of the European option is given by

\[
P^\epsilon(t, x) \overset{\epsilon \downarrow 0}{\sim} e^{-r(T-t)} \mathbb{E}^* \left[ h(\log(x + X_{T-t})) \right].
\]
Standardized Pseudo-Density

- **Pseudo-density** $p(t, - (s\sqrt{2\tilde{\sigma}^2}t \text{sign}(\rho) + \tilde{\mu}t)) = q(x; \Theta(t))$:

  $$q(x; \Theta) = \frac{1}{\Theta} \left[ Ai \left( \frac{\cdot}{\Theta} \right) \ast \mathcal{N}(\cdot) \right] (x), \quad \int_{-\infty}^{\infty} q(x; \Theta) \, dx = 1, \quad \Theta = \frac{3\sqrt{3v}}{\sqrt{2\tilde{\sigma}^2} \sqrt[4]{t}}.$$

- **Explicit form**:

  $$q(x; \Theta) = \exp \left( \frac{1}{12\Theta^6} + \frac{x}{2\Theta^3} \right) \Theta^{-1} Ai \left( \frac{x}{\Theta} + \frac{1}{4\Theta^4} \right),$$

- **Gaussian limit**: $q(x; \Theta) \xrightarrow{\Theta \to 0} \mathcal{N}(x)$. **Small volatility and times** gives relatively large support of Airy function and strong skew.

- **For smooth payoff function $\tilde{h}$ the support of Airy function relatively largest for $t^* = 1/\tilde{\sigma}^2$. Tails**:

  $$q(x; \Theta) \xrightarrow{x \to \infty} \frac{\exp \left( \frac{1}{12\Theta^6} \right)}{\left( \frac{4}{\sqrt{x}} \Theta^{3/4} \sqrt{2\sqrt{\pi}} \right)} \exp \left( \frac{-2 \left( x^{3/2} - 3x/(4\Theta^{3/2}) \right)}{3\Theta^{3/2}} \right),$$

  $$q(x; \Theta) \xrightarrow{x \to -\infty} \frac{\exp \left( \frac{1}{12\Theta^6} \right)}{\left( \frac{4}{\sqrt{x}} \Theta^{3/4} \sqrt{\pi} \right)} \exp \left( \frac{-|x|}{2\Theta^3} \right) \sin \left( \frac{2}{3} \left( \frac{x}{\Theta} + \frac{1}{4\Theta^4} \right) + \frac{\pi}{4} \right).$$
Pseudo-Density:
Airy dominated: $\Theta = 1 \ (v = 10^{-3}/3, \tilde{\sigma}^2 = .005, t = 1)$.

(Left) Transition Zone: $\Theta = .7 \ (v = 10^{-3}/3, \tilde{\sigma}^2 = .005, t \approx 3)$.
(Right) Gaussian Limit: $\Theta = .33 \ (v = 10^{-3}/3, \tilde{\sigma}^2 = .005, t \approx 700)$.

$\leftrightarrow$ Changing sign of correlation corresponds to time-reversal of the Airy function & pseudo-density.
We aim at \textbf{COMPUTING}:

\textbf{I: The joint survival probabilities:}

\[ q_n(T) = \mathbb{E}^* \left\{ e^{\int_0^T (\lambda_1 s + \ldots + \lambda_n s) \, ds} \right\} \]

for \( n = 1, \ldots, N \).

\textbf{II: The loss distribution:}

\[ p_n(T) = \mathbb{P}^* \{(\# \text{names defaulted by time } T) = n\} . \]

Stochastic Volatility Multiname Gaussian Model

• Choose **exchangeable** models and let the volatility be driven by common **fast mean reverting** SV factor:

\[
\begin{align*}
    dX_t^{(i)} &= \kappa(\theta - X_t^{(i)})dt + \sigma(Y_t)\, dW_t^{(i)}, \quad \text{with} \quad X_0^{(i)} = x, \\
    dY_t &= \frac{1}{\varepsilon}\alpha(Y_t)dt + \frac{1}{\sqrt{\varepsilon}}\beta(Y_t)\, dW_t^{(y)}, \quad \lambda_{it} = X_t^{(i)}.
\end{align*}
\]

where \(\varepsilon\) is the natural **time scale**/mean reversion time for the volatility factor.

• **name-name** correlations:

\[
    d\left\langle W^{(i)}, W^{(j)} \right\rangle_t = \rho_X \, dt, \quad i \neq j,
\]

& **name-volatility** correlations:

\[
    d\left\langle W^{(i)}, W^{(y)} \right\rangle_t = \rho_Y \, dt.
\]

↩ Negative intensities? Duffie-Singleton, Credit Risk 2003: “**the computational advantage with explicit solutions may be worth the approximation error associated with this Gaussian formulation**”.
The Corrected Multiname Survival Probability

For

\[ q_n(T) = \mathbb{E}\left\{ e^{- \int_0^T (X_s^{(1)} + \ldots + X_s^{(n)}) \, ds} \mid X_0^{(1)} = x, \ldots, X_0^{(n)} = x, Y_0 = y \right\} \]

we have in \( \varepsilon \rightarrow 0 \) limit:

\[ q_n(T) \sim \exp \left( \sqrt{\varepsilon \rho_Y v_3} B^{(3)}(T)(n^2 + n^2(n - 1)\rho_X) \right) \]

\[ \times \exp \left( -n \left[ \theta_\infty(T - B(T)) + [1 + (n - 1)\rho_X]\bar{\sigma}^2B^2(T)/(4\kappa) + xB(T) \right] \right) \]

with

\[ B(T) = \frac{1 - \exp(-\kappa T)}{\kappa}, \quad B^{(3)}(T) = \int_0^T B(t)^3 \, dt, \]

\[ \theta_\infty = \theta - [1 + (n - 1)\rho_X]\bar{\sigma}^2/2\kappa^2, \quad \bar{\sigma}^2 = \left\langle \sigma^2(y) \right\rangle. \]

\[ \rightarrow \text{Combined correlation gives strong correlation gearing.} \]
Leading Loss Distribution Symmetric Case

- **Loss distribution**: \( p_n = \binom{N}{n} \sum_{j=0}^{n} \binom{n}{j} q_{N+j-n}(-1)^j \), Conditioning:

  \[
  q_n = \mathbb{E}^* \{ e^{-d_1n+d_2nX} \}, \quad p_n = \mathbb{E}^* \{ \tilde{p}_n(d_1 - d_2X) \},
  \]

  for \( \tilde{p}_n(d) \) the binomial loss distribution with \( q_n = \exp(-dn) \) and \( X \) the pseudo-process \((v_3 > 0)\).

  \( \leftarrow \) Loss distribution is found via integration of binomial distribution with respect to an in general signed and non-symmetric measure. Gives loss distribution to \( \mathcal{O}(\varepsilon) \).

- **Name Heterogeneity**  
  **One volatility driving factor, heterogeneous level and volatility:** \( dX_{it} = \kappa(\theta_i - X^{(i)}_t)dt + v_if(Y_t)\,dW^{(i)}_t \), for \( 1 \leq i \leq N \) and with \( v_i \) and \( \theta_i \) constants.

  \( \leftarrow \) Constant volatility survival probability with name-name correlation and for \( X \) Gaussian:

  \[
  q(T; x, n) = \mathbb{E} \left\{ \prod_{i=1}^{n} e^{X^{\sigma_i} \sqrt{\rho X B^{(2)}(T)} \tilde{A}_i(T) e^{-B(T)} x_i} \right\} = \mathbb{E} \left\{ \prod_{i=1}^{n} \tilde{q}_i(T; x_i) \right\}
  \]

  \( \rightarrow \) Computation via Hull-White algorithm for independent case. \( \leftarrow \) General case \( X \) modeled as pseudo-variable.
CONCLUSION and FURTHER ISSUES

• Multiscale Stochastic Volatility provides flexible and parsimonious model class extensions that can be dealt with using singular and regular perturbation methods. Derives from a consistent framework honoring underlying price dynamics and provides a linkage between derivatives facilitating calibration.

• “Model independent” approach to dealing with uncertain and changing parameters.

• Situations with analytic constant parameter solutions in particular feasible.

• Generalizations to structural models, hedging schemes, indifference pricing, free boundary value problems, other markets ....

• Important issues: -“identifiability” of model parameters, efficient computations...