Risk Preferences
Further Developments beyond Random Variables

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(joint work with Michael Kupper)

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The intuitive notion of risk is very recent in history but remains unclear even today.

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LUHMANN
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_Luhmann_

In the context of economic theory, Knight gives a definition

_The practical difference between the two categories, risk and uncertainty, is that in the former the distribution of the outcome in a group of instances is known (either through calculation a priori or from statistic of past experience), while in the case of uncertainty this is not true._

_But Knight’s_ idea of “risk” does not match the one expressed in the theory of _monetary risk measures_ which typically address the risk of several probability models.
Motivation

The intuitive notion of risk is very recent in history but remains unclear even today.

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But Knight’s idea of “risk” does not match the one expressed in the theory of monetary risk measures which typically address the risk of several probability models.

Rather than in a descriptive way, we try to understand “risk” in a context (setting) independent manner, focusing on some crucial invariant features. These are

- “diversification should not increase the risk”
- “the better for sure, the less risky”
Outline

1. Risk Preferences – Robust Representation
2. Model Risk
3. Distributional Risk
4. Discounting Risk
5. Interplay Model Risk ↔ Distributional Risk
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Definitions

**Definition (Risk Order)**

A total preorder \( \succcurlyeq \) on \( \mathcal{X} \) is a risk order if it is

- **Quasiconvex**: \( x \succcurlyeq \lambda x + (1 - \lambda) y \) whenever \( x \succcurlyeq y \),
- **Monotone**: \( x \succcurlyeq y \) whenever \( y \succcurlyeq x \).

**Definition (Risk Measure)**

A function \( \rho : \mathcal{X} \rightarrow [-\infty, +\infty] \) is a risk measure if it is

- **Quasiconvex**: \( \rho(\lambda x + (1 - \lambda) y) \leq \max\{\rho(x), \rho(y)\} \),
- **Monotone**: \( \rho(x) \leq \rho(y) \) whenever \( x \succcurlyeq y \).

**Definition (Risk Acceptance Family)**

A family \( A = (A^m)_{m \in \mathbb{R}} \) of subset of \( \mathcal{X} \) is a risk acceptance family if it is

- **Convex**: \( A^m \) is convex,
- **Monotone**: \( A^m \subset A^n \) and \( x \succcurlyeq y \) for some \( y \in A^m \) implies \( x \in A^m \),
- **Right-Continuous**: \( A^m = \bigcap_{n > m} A^n \).
Here, $\mathcal{X}$ is a locally convex topological vector space with dual $\mathcal{X}^*$. $\triangleright$ is a vector preorder: $x \triangleright y$ iff $x - y \in \mathcal{K}$ closed convex cone with polar cone $\mathcal{K}^\circ$.

**Theorem (Robust Representation of l.s.c. Risk Orders)**

Any lower semicontinuous risk measure $\rho : \mathcal{X} \rightarrow [-\infty, +\infty]$ has a robust representation

$$\rho(x) = \sup_{x^* \in \mathcal{K}^\circ} R(x^*, \langle x^*, -x \rangle)$$

for a unique maximal risk function $R \in \mathcal{R}_{\text{max}}$.

**Definition**

$\mathcal{R}_{\text{max}}$ denotes the set of maximal risk functions $R : \mathcal{K}^\circ \times \mathbb{R} \rightarrow [-\infty, +\infty]$

- $R$ is jointly quasiconcave
- nondecreasing and left-continuous in the second argument
- $R(\lambda x^*, s) = R(x^*, s/\lambda)$ for any $\lambda > 0$
- $R$ has a uniform asymptotic minimum, $\lim_{s \rightarrow -\infty} R(x^*, s) = \lim_{s \rightarrow -\infty} R(y^*, s)$,
- $x^* \mapsto R^+(x^*, s) := \inf_{s' > s} R(x^*, s)$ is upper semicontinuous.
Possible settings by the specification of the convex set $\mathcal{X}$ and the monotonicity preorder $\succ$. 

- **Random variables** on $(\Omega, \mathcal{F}, P)$ with as preorder $\succ$ the “$\geq P$-almost surely”. 
- **Stochastic processes** modeling cumulative wealth processes $X = X_0, X_1, \ldots X_T$ with as preorder $\succ$ the cash flow monotonicity “$X_t - X_{t-1} := \Delta X_t \geq \Delta Y_t$”. 
- **Probability distributions** (lotteries) $\mathcal{M}_1$ is a convex set with standard monotonicity preorders $\succ$ either the first or second stochastic order. 
- **Cumulative consumption streams** are right continuous non decreasing functions $c : [0,1] \rightarrow \mathbb{R}^+$ building a convex cone. Here $c^{(1)}$ is “better for sure” than $c^{(2)}$ if $c^{(1)} - c^{(2)}$ is still a cumulative consumption stream. 
- **Stochastic kernels** are probability distributions subject to uncertainty, that is, mappings $\tilde{X} : \Omega \rightarrow \mathcal{M}_1$. Possible preorders $\succ$ are either the $P$-almost sure first or second stochastic order. 

...
Outline

1 Risk Preferences – Robust Representation

2 Model Risk

3 Distributional Risk

4 Discounting Risk

5 Interplay Model Risk ↔ Distributional Risk
Model Risk
Set of Random Variables

- \(X\) is the convex set \((A, B) := \{X \in \mathbb{L}_\infty | a < \text{ess inf } X \geq \text{ess sup } X < b\}\).
- \(\succ\): relation “greater than \(P\)-almost surely” corresponding to the cone \(K = \mathbb{L}_\infty^+\).
- Under the good topology \(\sigma (\mathbb{L}_\infty, \mathbb{L}_1)\), then \(K^\circ = \mathbb{L}_1^+\) and \(K_1^\circ = \{Z \in \mathbb{L}_1^+ | \mathbb{E}[Z] = 1\} =: \mathcal{M}_1(P)\) is a set of probability measures.

**Proposition (Random Variables \(\sim\) Modell Risk)**

Any \(\|\cdot\|_\infty\)-l.s.c. risk measure \(\rho : (A, B) \to [-\infty, +\infty]\) with the fatou property has a robust representation

\[
\rho(X) = \sup_{Q \in \mathcal{M}_1(P)} R(Q, \mathbb{E}_Q[-X])
\]

- Certainty equivalents of expected losses.
- Monotone Versions of Mean variance preferences (**Markowitz**)
- Coherent and convex monetary risk measures (**Artzner et al. and Föllmer and Schied**)
- Performance measures such as the **Sharpe** ratio and their monotone versions (**Cherny and Madan**)
- Economic index of riskiness (**Aumann and Serrano**)
- ...
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Distributional Risk
Set of Probability Distributions

- $\mathcal{X}$ is the convex set $\mathcal{M}_{1,c}$ of prob. dist. with compact support, i.e.,
  $\mu([-c, c]) = 1$ for some $c > 0$

- Diversification has a different meaning as for random variables: randomization.
  In general $P_{\lambda X + (1 - \lambda)Y} \neq \lambda P_X + (1 - \lambda) P_Y$.

- $\mu \succsim \nu$: first stochastic order, $\int l d\mu \geq \int l d\nu$ for any continuous increasing function $l$, or equivalently,

  $$F_\nu(x) := \nu([-\infty, x]) \geq \mu([-\infty, x]) =: F_\mu(x)$$
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Proposition (Probability Distributions $\leadsto$ Distributional Risk)

Any $\sigma(ca_c, C)$-l.s.c. risk measure $\rho: \mathcal{M}_{1,c} \to [-\infty, +\infty]$ monotone w.r.t. the first stochastic order has a robust representation

$$\rho(\mu) = \sup_{l \text{ continuous nondecreasing}} \left( R \left( l, - \int l(x) \mu(dx) \right) \right)$$
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However, in a recent paper with Freddy Delaben and Michael Kupper, we show that the monotonicity with respect to the first stochastic order allows to give a so called von Neumann and Morgenstern representation.

**Theorem (DDK 2010)**

Any affine risk measure $\rho$ of a risk order $\succeq$ on $M_1(\mathbb{R})$ (satisfies the archimedian and independance axiom) is $\sigma(M_1, B_b)$ continuous and can be represented by

$$\rho(\mu) = -\int l(x) \mu(dx)$$

for some nondecreasing bounded function $l$. 
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Distributional Risk

Example: Value at Risk

Probability Distributions: Value at Risk

\[ V_{\@R}q (X) = \sup \{ s \in \mathbb{R} \mid P [X + s \leq 0] \geq q \} \]

monotone and cash additive but not quasiconvex! Might even penalize diversification!

On the level of probability distribution

\[ V_{\@R}q (\mu) := \sup \{ s \in \mathbb{R} \mid \mu ([-\infty, -s]) \geq q \} \quad \text{note: } V_{\@R}q (P_X) = V_{\@R}q (X) \]

is a lower semi continuous risk measure for probability distributions as

\[ A^m = \{ \mu \mid V_{\@R}q (\mu) \leq m \} = \bigcap \{ \mu \mid \mu ([-\infty, s]) \geq q \} \quad \{ s > -m \} \]

It has a robust representation (only quasiconvex, not convex!)

\[ V_{\@R}q (\mu) = \sup_{\text{l continuous nondecreasing, } \inf l > -\infty} -l^{-1} \left( \frac{1}{1-q} \int l(x) \mu (dx) + \frac{q}{q-1} \inf l \right). \]
Distributional Risk
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**Distributional Risk**

Example: Value at Risk

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V@R_q (X) = \sup \left\{ s \in \mathbb{R} \left| P [X + s \leq 0] \geq q \right. \right\}
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Discounting Risk
Set of Consumption Streams

- $\mathcal{X}$ is the convex cone of (deterministic) consumption streams $c : [0, 1] \rightarrow \mathbb{R}^+$ (right continuous increasing).
- $c^1 \triangleright c^2$: if $c^1 - c^2$ is still a consumption stream.
- Some Orlicz topology (economically sound as argued by Hindy, Huang, Kreps).

**Proposition (Consumption Streams $\sim$ Discounting Risk)**

Any l.s.c. risk measure $\rho$ on the cone of consumption streams $c : [0, 1] \rightarrow \mathbb{R}^+$ has a robust representation

$$
\rho(c) = \sup_{\beta \in CS} R \left( \beta, - \int \beta(s) \, dc_s \right)
$$

Here, the $\beta$’s are some positive functions $\Rightarrow$ **discounting risk**.
Risk Preferences – Robust Representation Model Risk Distributional Risk Discounting Risk Interplay Model Risk ↔ Distributional Risk

Discounting Risk
Set of Consumption Streams

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Proposition (Consumption Streams $\sim$ Discounting Risk)

Any l.s.c. risk measure $\rho$ on the cone of consumption streams $c : [0, 1] \rightarrow \mathbb{R}^+$ has a robust representation

$$\rho(c) = \sup_{\beta \in CS^o} R\left(\beta, - \int \beta(s) \, dc_s\right)$$

Consumption Streams: Hindy-Huang-Kreps for exponential delay

$$\rho(c) := - \int_0^1 u\left(\int_0^t e^{-\gamma(t-s)} \, dc_s\right) = \sup_{\beta} \exp \left(- \int_0^1 \frac{\beta(s)}{\gamma} \, dc_s - \frac{g(\beta)}{\gamma}\right)$$

where

$$g(\beta) = \int_0^1 d\beta(s) - \gamma \int_0^1 \beta(s) \left[\frac{e^{-\gamma t}}{1 - e^{-\gamma t}} - \ln \left(\frac{\beta(t)}{1 - e^{-\gamma t}}\right) + \ln \left(\int_0^1 \frac{\beta(s)}{1 - e^{-\gamma s}} \, ds\right)\right] \, ds$$
This approach allows within one concept different interpretations of risk depending on the underlying context

**Random Variables** $\sim$ **Modell Risk**

$$\rho(X) = \sup_{Q} R(Q, E_Q[-X])$$

for probability models $Q$.

**Probability Distributions** $\sim$ **Distributional Risk**

$$\rho(\mu) = \sup_{l \text{ continuous nondecreasing}} R \left( l, - \int l(x) \mu(dx) \right)$$

for test functions $l$.

**Consumption Streams** $\sim$ **Discounting Risk**

$$\rho(c) = \sup_{\beta \in CS^\circ} R \left( \beta, - \int \beta(s) dc_s \right)$$

for discounting functions $\beta$. 

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Samuel Drapeau — Risk Preferences
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Interplay Model Risk $\leftrightarrow$ Distributional Risk

Stochastic kernels

Illustrate the interplay between model risk and distributional risk.

**Proposition**

$\rho$ is a l.s.c. risk measure of a risk order $\succcurlyeq$ on stochastic kernels $\tilde{X}(\omega, dx)$ monotone w.r.t. the $P$-almost sure second stochastic order and satisfying

$$\tilde{X}(\omega, \cdot) \succcurlyeq \tilde{Y}(\omega, dx) \text{ for any } \omega \in \Omega \implies \tilde{X} \succcurlyeq \tilde{Y}$$

Then, the risk order can be factorized into a model risk component and a distributional risk component, that is,

$$\rho\left(\tilde{X}\right) := \Phi\left(g\left(\tilde{X}(\omega, \cdot)\right)\right)$$

Where $\Phi$ is a l.s.c. risk measure on random variables $\mathbb{L}^\infty$ and $g$ is a risk measure on probability distributions $\mathcal{M}_{1,c}$. 
Temperature related long term insurance contract

Scenario dependant loss distribution of the contract: 
\[ \tilde{X}(\omega, dx) = \sum \mu_i 1_{\omega_i}(\omega). \]

\[ \rho(\tilde{X}) = \sup_{P=(p_1,p_2,...)} \left\{ \sum V@R_q(\mu_i)p_i - \alpha(P) \right\} \]
Conclusion

MANY THANKS FOR YOUR ATTENTION!