Comparative Analysis of VaR and Some Distortion Risk Measures

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1. The Class of Distortion Risk Measures (DRMs)

2. Statistical Estimation & Simulation

3. Capital Allocation with DRMs — simulation study
1. Distortion Risk Measures (DRM)

For a rv $X$ representing loss, put

- df of $X$: $F_X(x) := P(X \leq x)$
- quantile of $X$: $F_X^{-1}(u) := \inf \{ x \in \mathbb{R} : F_X(x) \geq u \}, \quad 0 < u < 1$

**Def:** A functional $\rho: L^\infty \rightarrow \mathbb{R}$ is called **coherent** if it satisfies

**[PO]** (positivity): $X \leq 0$ a.s. $\implies \rho(X) \leq 0$

**[PH]** (positive homogeneity): $\forall \lambda > 0, \quad \rho(\lambda X) = \lambda \rho(X)$

**[TE]** (translation equivariance): $\forall c > 0, \quad \rho(X + c) = \rho(X) + c$

**[SA]** (subadditivity): $\rho(X + Y) \leq \rho(X) + \rho(Y)$
Add two more axioms:

[LI] (law invariance): \[ X \overset{\mathcal{L}}{=} Y \implies \rho(X) = \rho(Y) \]

[CA] (comonotonic additivity):
\[
\text{if } X \text{ and } Y \text{ are comonotone} \implies \rho(X + Y) = \rho(X) + \rho(Y)
\]

\( X_1, \ldots, X_d \) are comonotone \( \iff \) There exist a rv \( Z \) and increasing func's \( f_1, \ldots, f_d \) s.t. \( (X_1, \ldots, X_d) \overset{\mathcal{L}}{=} (f_1(Z), \ldots, f_d(Z)) \)

Kusuoka: The class of DRMs coincides with the set of coherent risk measures satisfying law invariance and comonotonic additivity
Distortion function

Any distribution function (df) $D$ on $[0, 1]$;
i.e., right-continuous, increasing on $[0, 1]$, $D(0) = 0$, $D(1) = 1$

For a distortion $D$, a distortion risk measure (DRM) is defined by

$$
\rho_D(X) := \int_{[0,1]} F_X^{-1}(u) \, dD(u) = \int_{\mathbb{R}} x \, dD \circ F_X(x).
$$

[a.k.a. spectral risk measure (Acerbi), weighted V@R (Cherny)]

$\star$ $D^\text{VaR}_\alpha(u) = 1\{u \geq 1-\alpha\}$ yields $\text{VaR}_\alpha(X) = F_X^{-1}(1-\alpha)$, $0 < \alpha < 1$,
but this $D^\text{VaR}_\alpha$ is not convex.
**Example: Expected Shortfall (ES)**

The expected loss that is incurred when VaR is exceeded

\[
\text{ES}_\alpha(X) := \frac{1}{\alpha} \int_{1-\alpha}^{1} F_X^{-1}(u) \, du
\]

\[
= \mathbb{E}(X \mid X \geq \text{VaR}_\alpha(X))
\]

Taking distortion of the form

\[
D^{\text{ES}}_\alpha(u) = \frac{1}{\alpha} \left[ u - (1 - \alpha) \right]_+, \quad 0 < \alpha < 1
\]

yields ES as a DRM
Other Examples:

- **Proportional Hazards**: \( D_{\theta}^{\text{PH}}(u) = 1 - (1 - u)^\theta \)

- **Proportional Odds**: \( D_{\theta}^{\text{PO}}(u) = \frac{\theta u}{1 - (1 - \theta)u} \)

- **Gaussian (Wang transform)**: \( D_{\theta}^{\text{GA}}(u) = \Phi(\Phi^{-1}(u) + \log \theta) \)

- **Proportional \( \gamma \)-Odds**: \( D_{\theta}^{\text{PGO}}(u) = 1 - \left[ \frac{(1 - u)^\gamma}{\theta - \theta(1 - u)^\gamma + (1 - u)^\gamma} \right]^{1/\gamma} \)

- **Positive Poisson Mixture**: \( D_{\lambda}^{\text{PPM}}(u) = \frac{e^{\lambda u} - 1}{e^\lambda - 1} \)
2. Statistical Estimation

\((X_n)_{n \in \mathbb{N}}\): strictly stationary process with \(X_n \sim F\)

\(F_n\): empirical df based on the sample \(X_1, \ldots, X_n\)

A natural estimator of \(\rho(X)\) is

\[
\hat{\rho}_n(X) = \int_0^1 F_n^{-1}(u) dD(u)
\]

\[
= \sum_{i=1}^{n} c_{ni} X_{n:i}, \quad c_{ni} := D \left( \frac{i-1}{n}, \frac{i}{n} \right)
\]
**Strong consistency**

Let \( d(u) = \frac{d}{du} D(u) \) for a convex distortion \( D \), and \( 1 \leq p \leq \infty \), \( \frac{1}{p} + \frac{1}{q} = 1 \). Suppose that \((X_n)_{n \in \mathbb{N}}\) is an ergodic stationary sequence, and that \( d \in L^p(0, 1) \) and \( F^{-1} \in L^q(0, 1) \). Then

\[
\hat{\rho}_n(X) \longrightarrow \rho(X), \quad \text{a.s.}
\]

For a proof, see van Zwet (1980, AP)

[All we need is SLLN and Glivenko-Cantelli Theorem].
Assumptions:

- $(X_n)_{n \in \mathbb{N}}$ is strongly mixing with rate
  \[ \alpha(n) = O(n^{-\theta-\eta}) \text{ for some } \theta \geq 1 + \sqrt{2}, \eta > 0 \]

- For $F^{-1}$-almost all $u$, $d$ is continuous at $u$

- $|d| \leq B, \quad B(u) := Mu^{-b_1}(1-u)^{-b_2}$

- $|F^{-1}| \leq H, \quad H(u) := Mu^{-d_1}(1-u)^{-d_2}$

Assume $b_i, d_i$ & $\theta$ satisfy

\[ b_i + d_i + \frac{2b_i+1}{2\theta} < \frac{1}{2}, \quad i = 1, 2 \]
Set

\[
\sigma(u, v) := [u \wedge v - uv] + \sum_{j=1}^{\infty} [C_j(u, v) - uv] + \sum_{j=1}^{\infty} [C_j(v, u) - uv],
\]

\[
C_j(u, v) := \Pr(X_1 \leq F^{-1}(u), X_{j+1} \leq F^{-1}(v))
\]

**Theorem (Asymptotic Normality)**

Under the above assumptions, we have

\[
\sqrt{n}(\hat{\rho}_n(X) - \rho(X)) \xrightarrow{L} N(0, \sigma^2),
\]

where

\[
\sigma^2 := \int_0^1 \int_0^1 \sigma(u, v) d(u) d(v) dF^{-1}(u) dF^{-1}(v) < \infty
\]
• **GARCH model:**

\[
X_n = \sigma_n Z_n, \quad (Z_n) : \text{i.i.d.}
\]

\[
\sigma_n^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i X_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2
\]

- If the stationary distribution has a positive density around 0, then GARCH is strongly mixing with exponentially decaying \(\alpha(n)\)

• **Stochastic Volatility model:**

\[
X_n = \sigma_n Z_n, \quad (Z_n) : \text{i.i.d.,} \quad (\sigma_n) :\text{ strictly stationary positive}
\]

\( (Z_n) \) and \( (\sigma_n) \) are assumed to be independent

- The mixing rate of \((X_n)\) is the same as that of \((\log \sigma_n)\)
Simulation example: inverse-gamma SV model

\[ X_t = \sigma_t Z_t \]

\[ Z_t \text{ i.i.d. } N(0,1) \text{ and } V_t = 1/\sigma_t^2 \text{ satisfies} \]

\[ V_t = \rho V_{t-1} + \varepsilon_t, \]

where \( V_t \sim \text{Gamma}(a, b) \) for each \( t \), \( (\varepsilon_t) \text{ i.i.d. rv's} \), and \( 0 \leq \rho < 1 \)

\( \Rightarrow \) \( X_t \) has scaled \( t \)-distribution with \( \nu = 2a, \sigma^2 = b/a \)

Lawrance (1982): the distribution of \( \varepsilon_t \) is compound Poisson

Can be shown that \( (X_t) \) is geometrically ergodic
Simulation results for estimating VaR, ES & PO risk measures with inverse-gamma SV observations \((n = 500, \ # \ of \ replication = 1000)\)

\[ X_t = \sigma_t Z_t, \quad \text{where} \quad V_t = 1/\sigma_t^2 \quad \text{follows AR(1)} \]

with gamma(2,16000) marginal & \(\rho = 0.5\), \(Z_t\) i.i.d. \(N(0,1)\)

<table>
<thead>
<tr>
<th>(\theta = \alpha)</th>
<th>VaR bias</th>
<th>RMSE</th>
<th>ES bias</th>
<th>RMSE</th>
<th>PO bias</th>
<th>RMSE</th>
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<tr>
<td>0.1</td>
<td>0.0692</td>
<td>10.9303</td>
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<tr>
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</table>
Simulation results for estimating VaR, ES & PO risk measures with GARCH observations \((n = 500, \text{ # of replication } = 1000)\)

\[ X_t = 0.0009 + \varepsilon_t, \quad \sigma_t^2 = 0.5 + 0.85\sigma_{t-1}^2 + 0.1\varepsilon_{t-1}^2 \]

<table>
<thead>
<tr>
<th>(\theta = \alpha)</th>
<th>VaR</th>
<th>ES</th>
<th>PO</th>
<th>PH</th>
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</table>
• Estimation of Asymptotic Variance

\[
\sigma^2 = \int\int \sigma(F(x), F(y))d(F(x))d(F(y)) \, dx \, dy
\]

where

\[
\sigma(F(x), F(y)) = [F(x) \land F(y) - F(x)F(y)]
\]

\[
+ \sum_{j=1}^{\infty} [F_j(u, v) - F(x)F(y)] + \sum_{j=1}^{\infty} [F_j(y, x) - F(x)F(y)],
\]

and

\[
F_j(x, y) = P(X_1 \leq x, X_{j+1} \leq y)
\]

▶▶ How to estimate this? (to construct confidence intervals)
3. Capital Allocation

$d$ investment opportunities (e.g., business units, subportfolios, assets) $X_i$: loss associated with the $i$th investments

1. Compute the overall risk capital $\rho(X)$, where $X = \sum_{i=1}^{d} X_i$ and $\rho$ is a particular risk measure.

2. Allocate the capital $\rho(X)$ to the individual investment possibilities according to some mathematical capital allocation principle such that, if $\kappa_i$ denotes the capital allocated to the investment opportunity with potential loss $X_i$, we have $\sum_{i=1}^{d} \kappa_i = \rho(X)$.

Find $\kappa = (\kappa_1, \ldots, \kappa_d) \in \mathbb{R}^d$ s.t. $\sum_{i=1}^{d} \kappa_i = \rho(X)$ according to some criterion
Setup

It is convenient to introduce ‘weights’ \( \lambda = (\lambda_1, \ldots, \lambda_d) \)
(to be interpreted as amount of money invested in each opportunity)

Put \( X(\lambda) := \sum_{i=1}^{d} \lambda_i X_i \) and

\[ r_\rho(\lambda) := \rho(X(\lambda)) \quad \text{risk measure function} \]

If \( \rho \) is positive homogeneous, then, for \( h > 0 \)

\[ r_\rho(h\lambda) = hr_\rho(\lambda) \]

i.e., \( r_\rho \) is positive homogeneous of degree 1
Euler’s rule: If $r_\rho$ is positive homogeneous and differentiable,

$$r_\rho(\lambda) = \sum_{i=1}^{d} \lambda_i \frac{\partial r_\rho}{\partial \lambda_i}(\lambda)$$

**Euler allocation principle**

If $r_\rho$ is a positive homogeneous risk measure function, which is differentiable on the set $\Lambda$, then the (per-unit) Euler capital allocation principle associated with $r_\rho$ is

$$\kappa_i(\lambda) = \frac{\partial r_\rho}{\partial \lambda_i}(\lambda)$$
Justification

- Tasche: RORAC compatibility
  
  \( r_\rho \): differentiable risk measure function
  
  \( \kappa \): capital allocation principle
  
  \( \kappa \) is called *suitable for performance measurement* if for all \( \lambda \) we have

  \[
  \frac{\partial}{\partial \lambda_i} \left( \frac{-E(X(\lambda))}{r_\rho(\lambda)} \right) \begin{cases} 
  > 0 \quad & \text{if} \quad \frac{-E(X_i)}{\kappa_i(\lambda)} > \frac{-E(X(\lambda))}{r_\rho(\lambda)}, \\
  < 0 \quad & \text{if} \quad \frac{-E(X_i)}{\kappa_i(\lambda)} < \frac{-E(X(\lambda))}{r_\rho(\lambda)}. 
  \end{cases}
  \]

- The only per-unit capital allocation principle suitable for performance measurement is the Euler principle.
Denault: Coorperative game theory

$d$ investment opportunities $= d$ players

If $\rho$ is subadditive, then $\rho(X(\lambda)) \leq \sum_{i=1}^{d} \rho(\lambda_i X_i)$.

A fuzzy core (Aubin, 1981) is given by

$$
\mathcal{C} = \left\{ \kappa \in \mathbb{R}^d : r_\rho(1) = \sum_{i=1}^{d} \kappa_i \ & \ k & \ r_\rho(\lambda) \geq \sum_{i=1}^{d} \lambda_i \kappa_i \ \forall \lambda \in [0, 1]^d \right\}
$$

If $r_\rho$ is differentiable at $\lambda = 1$, then $\mathcal{C}$ consists only of the gradient vector of $r_\rho$ at $\lambda = 1$:

$$
\kappa_i = \left. \frac{\partial r_\rho(\lambda)}{\partial \lambda_i} \right|_{\lambda=1}
$$
Examples

- Covariance principle:

\[ r_\rho(\lambda) = \sqrt{\text{var}(X(\lambda))} = \sqrt{\lambda'\Sigma\lambda} \]

where \( \Sigma \) is the covariance matrix of \((X_1, \ldots, X_d)\). Then

\[ \kappa_i(\lambda) = \frac{\partial r_\rho(\lambda)}{\partial \lambda_i} = \frac{\text{cov}(X_i, X(\lambda))}{\sqrt{\text{var}(X(\lambda))}} \]

In particular, the capital allocated to the \( i \)th investment opportunity is

\[ \kappa_i = \frac{\text{cov}(X_i, X)}{\sqrt{\text{var}(X)}} \]
• VaR contributions:

\[ r_\rho(\lambda) = \text{VaR}_\alpha(X(\lambda)) \]

Then (Tasche, 1999)

\[ \kappa_i(\lambda) = \frac{\partial r_\rho(\lambda)}{\partial \lambda_i} = E(X_i \mid X(\lambda) = \text{VaR}_\alpha(X(\lambda))) \]

In particular, the capital allocated to the \( i \)th investment opportunity is given by

\[ \kappa_i = E(X_i \mid X = \text{VaR}_\alpha(X)) \]

(It is hard to compute, though)
• ES contributions:

\[ r_\rho(\lambda) = ES_\alpha(X(\lambda)) = \frac{1}{\alpha} \int_{1-\alpha}^{1} F_{X(\lambda)}^{-1}(u) \, du \]

Then

\[ \kappa_i(\lambda) = \frac{\partial r_\rho(\lambda)}{\partial \lambda_i} = E(X_i \mid X(\lambda) \geq \text{VaR}_\alpha(X(\lambda))) \]

In particular, the capital allocated to the \( i \)th investment opportunity is given by

\[ \kappa_i = E(X_i \mid X \geq \text{VaR}_\alpha(X)) \]
Capital Allocation with DRM

\[ r_\rho(\lambda) = \rho_D(X(\lambda)) = \int_{[0,1]} F_{X(\lambda)}^{-1}(u) \, dD(u) \]

Then, under some regularity conditions (Tsanakas),

\[ \kappa_i(\lambda) = \frac{\partial r_\rho(\lambda)}{\partial \lambda_i} = \int_{[0,1]} \frac{\partial}{\partial \lambda_i} F_{X(\lambda)}^{-1}(u) \, dD(u) \]

\[ = \int_{[0,1]} E[X_i | X(\lambda) = F_{X(\lambda)}^{-1}(u)] \, dD(u) \]

\[ = \int_{\mathbb{R}} E[X_i | X(\lambda) = x] d(F_{X(\lambda)}(x)) dF_{X(\lambda)}(x) \]

\[ = E[X_i d(F_{X(\lambda)}(X(\lambda)))] \]
Thus, the capital allocated to the $i$th investment opportunity is

$$\kappa_i = E[X_i d(F_X(X))]$$

We can think of $d(F_X(X))$ as a Radon-Nikodym density:

$$E(d(F_X(X))) = 1 \text{ trivially}$$

$$\frac{dQ}{dP} = d(F_X(X)) \implies \kappa_i = E^Q(X_i)$$

Even when we know the joint df of $(X_1, \ldots, X_d)$, it is still difficult to compute $\kappa_i$ since the joint df of $X_i$ and $X$ is needed (The only exception is a Gaussian case).

$\Rightarrow$ Resort to Monte Carlo
Given a random sample \((X^k_1, \ldots, X^k_d)\), \(k = 1, \ldots, n\), put

\[X^k = X^k_1 + \cdots + X^k_d, \quad \mathbb{F}_X(x) = \frac{1}{n + 1} \sum_{k=1}^{n} \mathbf{1}_{\{X_k \leq x\}}\]

Then we can estimate \(\kappa_i\) by

\[\hat{\kappa}_i = \frac{1}{n} \sum_{k=1}^{n} X^k_i d(\mathbb{F}_X(X^k)) \]

\[= \iint x_i d(\mathbb{F}_X(x)) d\mathbb{F}_{X_i,X}(x_i, x)\]

where

\[\mathbb{F}_{X_i,X}(x_i, x) = \frac{1}{n} \sum_{k=1}^{n} \mathbf{1}_{\{X_{k,i} \leq x_i, X^k \leq x\}}\]
The error $\hat{\kappa}_i - \kappa_i$ can be asymptotically evaluated by proving asymptotic normality: Under certain regularity conditions,

$$\sqrt{n}(\hat{\kappa}_i - \kappa_i) \xrightarrow{\mathcal{L}} N(0, \sigma^2)$$

where

$$\sigma^2 = \text{var} \left( F_{X_i}^{-1}(\xi_i) d(\xi) + \int\int F_{X_i}^{-1}(u_i) d'(u) 1_{\{\xi \leq u\}} \, dC_i(u_i, u) \right)$$

$$C_i(F_{X_i}(x_i), F_X(x)) = P(X_i \leq x_i, X \leq x) \text{ and } (\xi_i, \xi) \sim C_i$$

(Needs to be modified for ES)
**Numerical Experiments:** Take distortion densities

- **Expected Shortfall:**
  
  \[ d_\theta(u) = \frac{1}{\theta} \mathbf{1}_{\{u \geq 1 - \theta\}} \]

- **Proportional Odds:**
  \[ d_\theta(u) = \frac{\theta}{(1 - u + \theta u)^2} \]

- **Proportional Hazards:**
  \[ d_\theta(u) = \theta (1 - u)^{\theta - 1} \]

- **Gaussian:**
  \[ d_\theta(u) = \frac{\phi(\Phi^{-1}(u) + \log \theta)}{\phi(\Phi^{-1}(u))} \]
**Elliptical loss distribution:** $E_d(\mu, \Sigma, \psi)$

$\mu$: location vector, $\Sigma$: dispersion matrix, $\psi$: characteristic generator

Assume $r_\rho$ is the risk measure function of a positive homogeneous, law invariant risk measure $\rho$. Let $(X_1, \ldots, X_d) \sim E_d(0, \Sigma, \psi)$. Then under an Euler allocation, the relative capital allocation is given by

$$\frac{\kappa_i}{\kappa_j} = \frac{\kappa_i(1)}{\kappa_j(1)} = \frac{\sum_{k=1}^d \Sigma_{ik}}{\sum_{k=1}^d \Sigma_{jk}}, \quad 1 \leq i, j \leq d.$$

- The relative amounts of capital allocated to each investment opportunity are the same as long as we use a positive homogeneous, law invariant risk measure.
Estimated ratios $\hat{\kappa}_i / \hat{\kappa}_{i+1}$ of capital allocation ($\theta = \alpha = 0.05$)

sample from $N \left( \begin{pmatrix} 1 & 0.1 & 0.5 \\ 0.1 & 1 & 0.9 \\ 0.5 & 0.9 & 1 \end{pmatrix} \right)$, size $= n$, 1000 runs

| $n$ | true ratio | | bias | $\sqrt{\text{MSE}}$ | | bias | $\sqrt{\text{MSE}}$ | | bias | $\sqrt{\text{MSE}}$ | | bias | $\sqrt{\text{MSE}}$ |
|-----|------------|---|---------------------|---|---------------------|---|---------------------|---|---------------------|---|---------------------|
| 100 | 4/5        | ES | 0.0740              | 0.3962 | 0.0352              | 0.2815 | 0.0422              | 0.3281 | 0.0587              | 0.3933 |
|     |            |    |                     |        |                     |        |                     |        |                     |        |
|     | 5/6        |    | −0.0081              | 0.1045 | −0.0028              | 0.0793 | −0.0023              | 0.0908 | −0.0033              | 0.1048 |
| 250 | 4/5        | PO | 0.0129              | 0.2239 | 0.0101              | 0.1669 | 0.0219              | 0.2185 | 0.0332              | 0.2660 |
|     |            |    |                     |        |                     |        |                     |        |                     |        |
|     | 5/6        |    | 0.0007              | 0.0634 | −0.0003              | 0.0483 | −0.0017              | 0.0623 | −0.0030              | 0.0740 |
| 500 | 4/5        | PH | 0.0092              | 0.1441 | 0.0064              | 0.1103 | 0.0138              | 0.1594 | 0.0188              | 0.1911 |
|     |            |    |                     |        |                     |        |                     |        |                     |        |
|     | 5/6        |    | −0.0006              | 0.0429 | −0.0007              | 0.0329 | −0.0015              | 0.0465 | −0.0019              | 0.0552 |
| 5000| 4/5        | Gaussian | 0.0017              | 0.0459 | 0.0006              | 0.0356 | 10−5                | 0.0888 | 0.0005              | 0.0931 |
|     |            |    |                     |        |                     |        |                     |        |                     |        |
|     | 5/6        |    | −0.0003              | 0.0139 | 9·10−6              | 0.0108 | 0.0008              | 0.0265 | 0.0008              | 0.0278 |
Comparison in terms of DI ($\theta = \alpha = 0.05$)

Marginal: N(0,1)

Dependence: Gaussian & t copula with correlation matrix

\[
\begin{pmatrix}
1 & 0.1 & 0.5 \\
0.1 & 1 & 0.9 \\
0.5 & 0.9 & 1
\end{pmatrix}
\]

Compute diversification index: $\text{DI}_\rho(X) = \frac{\rho(X)}{\sum \rho(X_i)}$

- Gaussian: $\text{DI}_\rho(X) = 0.8165$ for all DRM $\rho$ theoretically
- t copula: $\text{DI}_{ES}(X) = 0.8329$ (std= 0.021), $\text{DI}_{PO}(X) = 0.8285$ (std= 0.015), $\text{DI}_{GA}(X) = 0.7367$ (std= 0.076)
Estimated capital allocation with GPD & t marginals ($\theta = \alpha = 0.05$) using Gaussian copula with correlation matrix

\[
\begin{pmatrix}
1 & 0.1 & 0.5 \\
0.1 & 1 & 0.9 \\
0.5 & 0.9 & 1
\end{pmatrix}
\]

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<th>PO</th>
<th>PH</th>
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<td>cont. ratio</td>
<td>cont. ratio</td>
<td>cont. ratio</td>
<td>cont. ratio</td>
</tr>
<tr>
<td>GPD(1/25)</td>
<td>2.60</td>
<td>2.21</td>
<td>1.58</td>
<td>3.25</td>
</tr>
<tr>
<td>GPD(1/10)</td>
<td>4.38 (0.59)</td>
<td>3.45 (0.64)</td>
<td>4.18 (0.38)</td>
<td>8.30 (0.39)</td>
</tr>
<tr>
<td>GPD(1/3)</td>
<td>9.12 (0.48)</td>
<td>6.99 (0.49)</td>
<td>24.32 (0.17)</td>
<td>38.87 (0.21)</td>
</tr>
<tr>
<td>t(25)</td>
<td>1.28</td>
<td>0.99</td>
<td>0.74</td>
<td>1.60</td>
</tr>
<tr>
<td>t(10)</td>
<td>2.04 (0.63)</td>
<td>1.54 (0.64)</td>
<td>1.69 (0.44)</td>
<td>3.44 (0.47)</td>
</tr>
<tr>
<td>t(3)</td>
<td>3.82 (0.53)</td>
<td>2.88 (0.54)</td>
<td>9.62 (0.18)</td>
<td>14.97 (0.23)</td>
</tr>
</tbody>
</table>
4. Concluding Remarks

- Estimation of DRMs is possible, but for some DRMs, we don’t get nice asymptotic properties; proportional odds risk measure has some nice features.

- Euler capital allocation based on DRMs are easy to compute and widely applicable (more stable than VaR). Need more computational efficiency for tail-exaggerating DRMs.

- Future research: Careful study of portfolio optimization

- Future research: Extension to dynamic setting