Efficient Risk Estimation via Nested Sequential Simulation

Mark Broadie, Columbia University

Joint work with Yiping Du and Ciamac Moallemi

Bachelier Finance Society

June 24, 2010
Risk Measurement

Security positions today

- Hundreds or thousands of securities
- Stocks, bonds, options, swaps, structured products
- Equities, fixed income, foreign exchange, commodities

Security values at risk horizon $\tau$

- Multiple underlying financial factors
- Financial model: distribution of factors at $\tau$
- Security prices at $\tau$ in state $\omega$
- Prices depend on cashflows from time $\tau$ to $T$
- Distribution of portfolio losses $L(\omega)$

Risk measure

- Distribution of losses $L(\omega)$ is mapped to a risk measure $\rho(L)$
The Risk Measurement Problem

- Today: $t = 0$

- Risk horizon: $t = \tau$

- $\omega$ = state at time $\tau$

- $L(\omega)$ = portfolio loss at time $\tau$, given state $\omega$

- $L(\omega)$ depends on realized cashflows between $\tau$ and $T$

- Risk measure $\rho(L)$ ∈ $\mathbb{R}$

- Probability of large loss: $P(L \geq c)$

- $\text{VAR}_\alpha(L) = \inf \left\{ c : P(L \geq c) \leq \alpha \right\}$

- $\text{CVAR}_\alpha(L) = E[L | L \geq \text{VAR}_\alpha(L)]$

Broadie, Du and Moallemi: *Risk Estimation via Nested Sequential Simulation*
The Risk Measurement Problem

- Today: $t = 0$
- Risk horizon: $t = \tau$

\[ L(\omega) = \text{portfolio loss at time } \tau, \text{ given state } \omega \]
The Risk Measurement Problem

- Today: \( t = 0 \)
- Risk horizon: \( t = \tau \)

\[ \omega = \text{state at time } \tau \]

\[ L(\omega) = \text{portfolio loss at time } \tau, \text{ given state } \omega \]

- \( L(\omega) \) depends on realized cashflows between \( \tau \) and \( T \)
The Risk Measurement Problem

- Today: $t = 0$
- Risk horizon: $t = \tau$

$\omega =$ state at time $\tau$

$L(\omega) =$ portfolio loss at time $\tau$, given state $\omega$

- $L(\omega)$ depends on realized cashflows between $\tau$ and $T$
- Risk measure $\rho(L) \in \mathbb{R}$

Probability of large loss: $P(L \geq c)$

$\text{VAR}_\alpha(L) = \inf \{c \mid P(L \geq c) \leq \alpha\}$

$\text{CVAR}_\alpha(L) = \mathbb{E}[L \mid L \geq \text{VAR}_\alpha(L)]$

Coherent risk measures ...
Related Literature

- Uniform nested simulation
  - Lee (1998)
  - Gordy and Juneja (2006, 2008)

- Importance sampling
  - Glasserman, Heidelberger, Shahabuddin (2000)

- Stochastic kriging
  - Liu and Staum (2009)
The Risk Measurement Problem

- Simulate $\omega_1, \ldots, \omega_n$
The Risk Measurement Problem

- Simulate $\omega_1, \ldots, \omega_n$
- For each $\omega_i$: simulate future portfolio cashflows $\hat{Z}_{i,1}, \ldots, \hat{Z}_{i,m}$

\[
\hat{L}_i = \frac{1}{m} \sum_{j=1}^{m} \hat{Z}_{i,j}
\]

estimate of loss $L(\omega_i)$
• Simulate $\omega_1, \ldots, \omega_n$

• For each $\omega_i$: simulate future portfolio cashflows $\hat{Z}_{i,1}, \ldots, \hat{Z}_{i,m}$

$\hat{L}_i = \frac{1}{m} \sum_{j=1}^{m} \hat{Z}_{i,j}$ \hspace{1cm} \text{estimate of loss $L(\omega_i)$}

• Estimate probability of loss

$\hat{\alpha} = \frac{1}{n} \sum_{i=1}^{n} 1\{\hat{L}_i \geq c\}$
Probability of Loss: Gaussian Example

- First stage: \( L(\omega_i) = \omega_i \), where \( \omega_i \sim N(0, \sigma_1^2) \)

- Second stage: \( Z_{i,j} = \omega_i + \epsilon_{i,j} \), where \( \epsilon_{i,j} \sim N(0, \sigma_2^2) \)

- Probability of loss: \( \alpha = P(L \geq c) = \Phi(-c/\sigma_1) \)

Estimator: \( \hat{\alpha} = \frac{1}{n} \sum_{i=1}^{n} 1\{\hat{L}_i \geq c\} \) where \( \hat{L}_i = L_i + \frac{1}{m} \sum_{j=1}^{m} \hat{Z}_{i,j} \)

Mean-Squared Error (MSE):

\[
\text{MSE} = E[(\hat{\alpha} - \alpha)^2]
= E[(\hat{\alpha} - E(\hat{\alpha}))^2] + (E[\hat{\alpha} - \alpha])^2
= \text{Variance} + \text{Bias}^2
\]
For $L_i > c$, $\mathbf{1}_{\{L_i \geq c\}} = 1$, but $E[\mathbf{1}_{\{\hat{L}_i \geq c\}}] = P(\hat{L}_i \geq c) < 1$. The local bias is negative: $E[\mathbf{1}_{\{\hat{L}_i \geq c\}} - 1] < 0$. 

Broadie, Du and Moallemi: *Risk Estimation via Nested Sequential Simulation*
For $L_i < c$, $\mathbf{1}\{L_i \geq c\} = 0$, but $E[\mathbf{1}\{\hat{L}_i \geq c\}] = P(\hat{L}_i \geq c) > 0$. The local bias is positive: $E[\mathbf{1}\{\hat{L}_i \geq c\} - 0] > 0$. 

Bias Illustration
Optimal MSE Formulation

\[ \omega_1, \omega_2, \ldots, \omega_i, \ldots, \omega_n \]

\[ \hat{Z}_{i,1}, \ldots, \hat{Z}_{i,m} \]

Time

\( t \)

0 \quad \tau \quad T

\( n \) first stage samples
Optimal MSE Formulation

\[ \omega_1, \omega_2, \ldots, \omega_i, \ldots, \omega_n \]

\[ \hat{Z}_{i,1}, \hat{Z}_{i,2}, \ldots, \hat{Z}_{i,m} \]

\begin{itemize}
  \item \( n \) first stage samples
  \item \( m \) second stage samples
\end{itemize}

Broadie, Du and Moallemi: *Risk Estimation via Nested Sequential Simulation*
Optimal MSE Formulation

\[ \omega_1 \quad \omega_2 \quad \ldots \quad \omega_i \quad \ldots \quad \omega_n \]

\[ \hat{Z}_{i,1} \quad \hat{Z}_{i,2} \quad \ldots \]

\[ 0 \quad \tau \quad T \]

\( n \) first stage samples \quad \( m \) second stage samples

total work: \( k = mn \)
Optimal MSE Formulation

\[ \omega_1, \omega_2, \ldots, \omega_i, \ldots, \omega_n \]

\[ \hat{Z}_{i,1}, \hat{Z}_{i,2}, \ldots, \hat{Z}_{i,m} \]

Time $t$

0 $\tau$ $T$

$n$ first stage samples $m$ second stage samples

total work: $k = mn$

Optimal allocation problem:

\[ \text{minimize }_{n,m} \text{ MSE} \]

subject to $nm = k$
Bias and Variance

\[ \alpha = P(L \geq c) \quad \hat{\alpha} = \frac{1}{n} \sum_{i=1}^{n} 1\{\hat{L}_i \geq c\} \]

\[ \text{MSE} = \mathbb{E} \left[ (\hat{\alpha} - \mathbb{E}\hat{\alpha})^2 \right] + \left( \mathbb{E}[\alpha - \hat{\alpha}] \right)^2 \]

\( \text{variance} + \text{bias}^2 \)

Under mild technical assumptions, as \( m, n \uparrow \infty \):

\[ \text{variance} \rightarrow \alpha(1 - \alpha) n \] \[ \text{bias} \rightarrow \gamma m \]

Optimal allocation:

\[ \minimize_{n, m} \text{MSE} \quad \text{subject to} \quad nm = k \]

\[ \begin{align*}
    n^* &= Ck^2/3 \\
    m^* &= 1/Ck^{1/3}
\end{align*} \]

\[ \text{MSE} \propto k^{-2/3} \]

Gordy and Juneja (2006)

Broadie, Du and Moallemi: Risk Estimation via Nested Sequential Simulation
\( \alpha = P(L \geq c) \)  
\( \hat{\alpha} = \frac{1}{n} \sum_{i=1}^{n} 1\{\hat{L}_i \geq c\} \)

\[
MSE = E\left[ (\hat{\alpha} - E\hat{\alpha})^2 \right] + \left( E[\alpha - \hat{\alpha}] \right)^2
\]

\[\text{variance} + \text{bias}^2\]

Under mild technical assumptions, as \( m, n \uparrow \infty \):

variance \( \rightarrow \frac{\alpha(1 - \alpha)}{n} \)  
bias \( \rightarrow \frac{\gamma}{m} \)
Bias and Variance

\[ \alpha = P(L \geq c) \quad \hat{\alpha} = \frac{1}{n} \sum_{i=1}^{n} 1\{\hat{L}_i \geq c\} \]

\[ \text{MSE} = E \left[ (\hat{\alpha} - E\hat{\alpha})^2 \right] + \left( E[\alpha - \hat{\alpha}] \right)^2 \]

\[
\text{variance} \rightarrow \frac{\alpha(1 - \alpha)}{n} \quad \text{bias} \rightarrow \frac{\gamma}{m}
\]

Under mild technical assumptions, as \( m, n \uparrow \infty \):

\[
\text{variance} \rightarrow \frac{\alpha(1 - \alpha)}{n} \quad \text{bias} \rightarrow \frac{\gamma}{m}
\]

Optimal allocation:

\[
\text{minimize } \text{MSE} \quad \text{subject to } nm = k \quad \Rightarrow \quad \begin{cases} 
  n^* = Ck^{2/3} \\
  m^* = \frac{1}{C}k^{1/3} \\
  \text{MSE} \propto k^{-2/3}
\end{cases}
\]

Gordy and Juneja (2006)
Optimal MSE Estimator

Optimal allocation: $n^* = Ck^{2/3}$, $m^* = \frac{1}{C}k^{1/3}$, $\text{MSE} \propto k^{-2/3}$
Optimal allocation: $n^* = Ck^{2/3}$, $m^* = \frac{1}{C} k^{1/3}$, $\text{MSE} \propto k^{-2/3}$

Observations:
- Similar expressions for VAR and CVAR, different constants
Optimal allocation: \( n^* = C k^{2/3}, \ m^* = \frac{1}{C} k^{1/3}, \ \text{MSE} \propto k^{-2/3} \)

Observations:

- Similar expressions for VAR and CVAR, different constants
- Not clear how to implement! Need to estimate the constant \( C \)
Optimal MSE Estimator

Optimal allocation: $n^* = Ck^{2/3}$, $m^* = \frac{1}{C}k^{1/3}$, $\text{MSE} \propto k^{-2/3}$

Observations:

- Similar expressions for VAR and CVAR, different constants
- Not clear how to implement! Need to estimate the constant $C$
- Can we do better?
Non-Uniform Sampling

Idea: use a non-uniform number of stage 2 samples

\[ m_i = \text{number of samples at } \omega_i \]
Idea: use a non-uniform number of stage 2 samples

\[ m_i = \text{number of samples at } \omega_i \]

\[ \hat{\alpha} = \frac{1}{n} \sum_{i=1}^{n} 1\{\hat{L}_i \geq c\} \]
Non-Uniform Sampling

Idea: use a non-uniform number of stage 2 samples

\[ m_i = \text{number of samples at } \omega_i \]

\[ \hat{\alpha} = \frac{1}{n} \sum_{i=1}^{n} 1\{\hat{L}_i \geq c\} \]
Non-Uniform Sampling

Idea: use a non-uniform number of stage 2 samples

\[ m_i = \text{number of samples at } \omega_i \]

\[ \hat{\alpha} = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{\{\hat{L}_i \geq c\}} \]

Broadie, Du and Moallemi: Risk Estimation via Nested Sequential Simulation
Non-Uniform Sampling

Idea: use a non-uniform number of stage 2 samples

\[ m_i = \text{number of samples at } \omega_i \]

\[ \hat{\alpha} = \frac{1}{n} \sum_{i=1}^{n} 1\{\hat{L}_i \geq c\} \]

Broadie, Du and Moallemi: Risk Estimation via Nested Sequential Simulation
Non-Uniform Sampling

Idea: use a non-uniform number of stage 2 samples

\[ m_i = \text{number of samples at } \omega_i \]

\[ \hat{\alpha} = \frac{1}{n} \sum_{i=1}^{n} 1\{\hat{L}_i \geq c\} \]

Broadie, Du and Moallemi: *Risk Estimation via Nested Sequential Simulation*
Non-Uniform Sampling

Idea: use a non-uniform number of stage 2 samples

\[ m_i = \text{number of samples at } \omega_i \]

\[ \hat{\alpha} = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{\{\hat{L}_i \geq c\}} \]

\[ \hat{Z}_{i,1}, \ldots, \hat{Z}_{i,m_i} \]

\[ \omega_1, \omega_2, \ldots, \omega_i, \ldots, \omega_n \]

Broadie, Du and Moallemi: Risk Estimation via Nested Sequential Simulation
Stage 2 Algorithm

\[ \hat{L}_i = \frac{1}{m_i} \sum_{j=1}^{m_i} \hat{Z}_{i,j} \]

\[ \hat{\alpha} = \frac{1}{n} \sum_{i=1}^{n} 1\{\hat{L}_i \geq c\} \]

Idea:

- Sequentially add stage 2 samples

Broadie, Du and Moallemi: *Risk Estimation via Nested Sequential Simulation*
Stage 2 Algorithm

Idea:

- Sequentially add stage 2 samples
- Add the next sample where it will most affect the estimate $\hat{\alpha}$

Mathematical Formulas:

$$
\hat{L}_i = \frac{1}{m_i} \sum_{j=1}^{m_i} \hat{Z}_{i,j}
$$

$$
\hat{\alpha} = \frac{1}{n} \sum_{i=1}^{n} 1\{\hat{L}_i \geq c\}
$$
Stage 2 Algorithm

Idea:

- Sequentially add stage 2 samples
- Add the next sample where it will most affect the estimate $\hat{\alpha}$

Mathematical Expressions:

$\hat{L}_i = \frac{1}{m_i} \sum_{j=1}^{m_i} \hat{Z}_{i,j}$

$\hat{\alpha} = \frac{1}{n} \sum_{i=1}^{n} 1\{\hat{L}_i \geq c\}$
Stage 2 Algorithm

Probability

Loss

\[ \hat{L}_i = \frac{1}{m_i} \sum_{j=1}^{m_i} \hat{Z}_{i,j} \]

\[ \hat{\alpha} = \frac{1}{n} \sum_{i=1}^{n} 1\{\hat{L}_i \geq c\} \]

Idea:

- Sequentially add stage 2 samples
- Add the next sample where it will most affect the estimate \( \hat{\alpha} \)
Stage 2 Algorithm

\[
\hat{L}_i = \frac{1}{m_i} \sum_{j=1}^{m_i} \hat{Z}_{i,j}
\]

\[
\hat{\alpha} = \frac{1}{n} \sum_{i=1}^{n} 1\{\hat{L}_i \geq c\}
\]

Idea:

- Sequentially add stage 2 samples
- Add the next sample where it will most affect the estimate \( \hat{\alpha} \)
- Use a normal approximation: given one more sample at \( \omega_i \),

\[
P(\text{estimate } \hat{\alpha} \text{ changes}) \approx \Phi \left( -\frac{m_i}{\sigma^2} | \hat{L}_i - c | \right)
\]
Non-Uniform Stage 2 Algorithm

- Simulate $\omega_1, \ldots, \omega_n$
- For each $\ell$ from 1 to $k$:
  - Pick $i^* \in \text{argmin}_i \frac{m_i}{\sigma^2} \left| \hat{L}_i - c \right|$, Add 1 sample at $\omega_{i^*}$
- Estimate probability of loss
  $$\hat{\alpha} = \frac{1}{n} \sum_{i=1}^{n} 1\{\hat{L}_i \geq c\}$$
Key Result

Under suitable assumptions,

\[
\text{bias} \propto \frac{1}{\bar{m}^2} \quad \left(\text{vs. bias} \propto \frac{1}{m} \text{ under uniform sampling}\right)
\]
Under suitable assumptions,
\[
\text{bias } \propto \frac{1}{\bar{m}^2} \quad \text{ (vs. bias } \propto \frac{1}{m} \text{ under uniform sampling)}
\]

**Proof Technique:**
For a given \( \omega_i \), consider the sequential hypothesis testing problem:

- Observe IID samples \( \hat{Z}_{i,1}, \hat{Z}_{i,2}, \ldots \) with \( L(\omega_i) = \mathbb{E}[Z_{i,1}] \)
- Hypotheses:
  \[
  H_0(\omega_i) = \{ L(\omega_i) < c \}
  
  H_1(\omega_i) = \{ L(\omega_i) \geq c \}
  
  \]
- We wish to determine which hypothesis is true, with a minimal number of observations

Our non-uniform sampling algorithm is solving many sequential hypothesis testing problems simultaneously.
Rate of Convergence

- Uniform algorithm:
  
  \[
  \begin{aligned}
  &\text{minimize} & n, m \quad \text{MSE} \\
  \text{subject to} & \quad nm = k
  \end{aligned}
  \Rightarrow \begin{cases}
  n^* \propto k^{2/3} \\
  m^* \propto k^{1/3} \\
  \text{MSE} \propto k^{-2/3}
  \end{cases}
  \]

- Non-uniform algorithm:
  
  \[
  \begin{aligned}
  &\text{minimize} & n, \bar{m} \quad \text{MSE} \\
  \text{subject to} & \quad n\bar{m} = k
  \end{aligned}
  \Rightarrow \begin{cases}
  n^* \propto k^{4/5} \\
  \bar{m}^* \propto k^{1/5} \\
  \text{MSE} \propto k^{-4/5}
  \end{cases}
  \]
Gaussian Example

First stage: \( L(\omega_i) = \omega_i \), where \( \omega_i \sim N(0, \sigma_1^2) \)

Second stage: \( Z_{i,j} = \omega_i + \epsilon_{i,j} \), where \( \epsilon_{i,j} \sim N(0, \sigma_2^2) \)

Probability of loss: \( \Pr(L \geq c) = \Phi(-c/\sigma_1) \)
Number of Inner Stage Samples versus Loss

\[ c = 2.326 \]

**Broadie, Du and Moallemi:** Risk Estimation via Nested Sequential Simulation
Bias versus Number of Inner Stage Samples

\[ \text{Bias} \propto k^{-2} \]

\[ \text{Bias} \propto k^{-1} \]

Total number of inner stage samples \( k \)

- Sequential
- Uniform

Broadie, Du and Moallemi: Risk Estimation via Nested Sequential Simulation
Numerical Results: Gaussian Example

\[ \sigma_1 = 1, \quad \sigma_2 = 5, \quad \alpha = 0.1\%, \quad k = 4,000,000 \]

<table>
<thead>
<tr>
<th>Method</th>
<th>( n )</th>
<th>( \bar{m} )</th>
<th>MSE</th>
<th>Rel MSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n = m = \sqrt{k} )</td>
<td>2,000</td>
<td>2,000</td>
<td>( 5.7 \cdot 10^{-7} )</td>
<td>23</td>
</tr>
<tr>
<td>( n = k^{2/3}, \quad m = k^{1/3} )</td>
<td>25,200</td>
<td>159</td>
<td>( 1.2 \cdot 10^{-6} )</td>
<td>48</td>
</tr>
<tr>
<td>uniform (optimal constant)</td>
<td>7,788</td>
<td>514</td>
<td>( 2.5 \cdot 10^{-7} )</td>
<td>10</td>
</tr>
<tr>
<td>adaptive</td>
<td>30,628</td>
<td>132</td>
<td>( 3.6 \cdot 10^{-8} )</td>
<td>1.5</td>
</tr>
<tr>
<td>optimal sequential</td>
<td>56,686</td>
<td>71</td>
<td>( 2.5 \cdot 10^{-8} )</td>
<td>1</td>
</tr>
</tbody>
</table>
Put Option Example

- **Stock price:** \( S_T(\omega) \triangleq S_0 e^{(\mu - \sigma^2/2)\tau + \sigma \sqrt{\tau} \omega} \)
- **\( L(\omega) = X_0 - \mathbb{E} \left[ e^{-r(T-\tau)} \max (K - S_T(\omega, W), 0) \mid \omega \right] \)** where
  \[
  S_T(\omega, W) \triangleq S_\tau(\omega) e^{(r-\sigma^2/2)(T-\tau) + \sigma \sqrt{T-\tau} W}
  \]
  and
  \[
  \hat{Z}_{i,j} = X_0 - e^{-r(T-\tau)} \max \left( K - S_T(\omega_i, W_{i,j}), 0 \right),
  \]
- **Outer stage:** the real-world distribution (\( \mu \))
- **Inner stage:** risk-neutral distribution (\( r \))
Numerical Results: Put Option

\[ S_0 = 100, \; K = 95, \; \sigma = 20\%, \; \tau = 1/52, \; T = 0.25 \]
\[ \alpha = 0.1\%, \; k = 4,000,000 \]

<table>
<thead>
<tr>
<th>Method</th>
<th>( n )</th>
<th>( \tilde{m} )</th>
<th>MSE</th>
<th>Rel MSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n = m = \sqrt{k} )</td>
<td>2,000</td>
<td>2,000</td>
<td>( 5.6 \cdot 10^{-7} )</td>
<td>12</td>
</tr>
<tr>
<td>( n = k^{2/3}, ; m = k^{1/3} )</td>
<td>25,200</td>
<td>159</td>
<td>( 8.2 \cdot 10^{-6} )</td>
<td>175</td>
</tr>
<tr>
<td>uniform (optimal constant)</td>
<td>2,570</td>
<td>1,556</td>
<td>( 4.8 \cdot 10^{-7} )</td>
<td>10</td>
</tr>
<tr>
<td>adaptive</td>
<td>14,384</td>
<td>284</td>
<td>( 9.2 \cdot 10^{-8} )</td>
<td>2</td>
</tr>
<tr>
<td>optimal sequential</td>
<td>26,508</td>
<td>151</td>
<td>( 4.7 \cdot 10^{-8} )</td>
<td>1</td>
</tr>
</tbody>
</table>
Summary

- Nested simulation can provide a more realistic assessment of risk
- Reduced computational burden by
  - Non-uniform inner sampling to reduce bias
  - More outer sampling to reduce variance
- MSE reduced by factors from 4 to over 100