Hedging under Model Uncertainty

Efficient Computation of the Hedging Error using the POD

6th World Congress of the Bachelier Finance Society
June, 24th 2010

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Ulm University

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Hedging under Model Uncertainty

Considered Models

Reduced Model and POD

Results
Hedging under Model Uncertainty

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Reduced Model and POD

Results
Hedging a exotic Option under Model Uncertainty

Exotic Option: Asian Call

\[ C^A(T, S_T) = (\bar{S}_T - K)^+ \]

- Model uncertainty: “true model” \( \neq \) hedge model
- Relation between “true model” and hedge model:
  - Vanilla Options for the calibration of the hedge model parameters
Hedging under Model Uncertainty

Hedging a exotic Option under Model Uncertainty

Exotic Option: Asian Call

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- Model uncertainty: “true model” ≠ hedge model
- Relation between “true model” and hedge model:
  - Vanilla Options for the calibration of the hedge model parameters
- Hedging Approach:
  - Delta-Hedge with bank account and underlying
  - Delta- and Vega-Hedge with bank account, underlying and vanilla option
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Results
“True Models”

3-Factor Model (3FM)

\[(S_t, \nu_t, \rho_t) \text{ driven by } (W_1^t, W_2^t, W_3^t)\]

Extended Heston Model with correlated jumps (SVJ)

\[S_t = S_0 e^{x_t} \text{ where}\]

\[dx_t = \left(\mu - \frac{1}{2} \nu_t\right) dt + \sqrt{\nu_t} dW_1^t + \xi^x dN_t\]

\[dv_t = \alpha (\beta - \nu_t) dt + \sigma \sqrt{\nu_t} \left(\rho dW_1^t + \sqrt{1 - \rho^2} dW_2^t\right) + \xi^v dN_t\]

Extended CGMY Model (CGMYe)

\[S_t = S_0 \exp \left\{ (\mu + \omega - \eta^2/2) t + X_{CGMY}^t + \eta W_t \right\}\]
Hedge Models

Black-Scholes Model (BS)

\[ dS_t = rS_t dt + \sigma S_t dW_t \]

Stochastic Alpha, Beta, Rho Model (SABR)

\[ dS_t = rS_t dt + \sigma_t S_t dW_t^1 \]
\[ d\sigma_t = \alpha \sigma_t \left( \rho dW_t^1 + \sqrt{1 - \rho^2} dW_t^2 \right) \]

Heston Model (SV)

\[ dS_t = rS_t dt + \sqrt{v_t} S_t dW_t^1 \]
\[ dv_t = \alpha (\beta - v_t) dt + \sigma_v \sqrt{v_t} \left( \rho dW_t^1 + \sqrt{1 - \rho^2} dW_t^2 \right) \]
## Model Properties

### “True Models”
- Models represent various “stylized facts”.
- Calibrated parameter are available for $\mathbb{P}$ and for $\mathbb{Q}$.
- Vanilla Option prices available via
  - Monte-Carlo (3FM)
  - Analytic Formula (SVJJ)
  - PIDE (CGMYe)
- Valuation of the Asian Option by Monte-Carlo.

## Hedge Models
- (Semi-)analytic formulas available for the Vanilla Option prices.
- Valuation of the Asian Option and hedging weights via PDE.
Hedging under Model Uncertainty

Considered Models

Reduced Model and POD

Results
PDE in Heston Model

Idea: Vecer ’02, Shreve ’08 \( \Rightarrow C^A(t, S_t, v_t) = S_t g(t, Y_t, v_t) \) where

2-dimensional parabolic PDE in Heston Model

\[
\frac{\partial}{\partial t} g(t, y, v) + \nu \left( \tilde{\phi} - v \right) \frac{\partial}{\partial v} g(t, y, v) + \frac{1}{2} \nu (q_t - y)^2 \frac{\partial^2}{\partial y^2} g(t, y, v) \\
+ \frac{1}{2} \varphi^2 v \frac{\partial^2}{\partial v^2} g(t, y, v) + \varphi \rho v (q_t - y) \frac{\partial^2}{\partial y \partial v} g(t, y, v) = 0,
\]

\( g(T, y, v) = y^+ \)

and

\[
Y_t = \frac{1}{S_t} \left( \frac{1}{T} \int_0^t S_u du - K \right) \quad \text{on } \Omega = (-1, 1) \times (0, \infty).
\]

Further Advantage: Greeks are directly computable from the PDE.
PDE in Black-Scholes Model

Again we have $C^A(t, S_t) = S_t g(t, Y_t)$ where this time

1-dimensional parabolic PDE in Black-Scholes Model

$$\frac{\partial}{\partial t} g(t, y) - \frac{\sigma^2}{2} (q_t - y)^2 \frac{\partial^2}{\partial y^2} g(t, y) = 0,$$

$g(0, y) = y^+$ on $\Omega = (-1, 1)$

Similar PDE for $\Lambda$ (Vega)

$$\frac{\partial}{\partial t} \Lambda(t, y) - \frac{\sigma^2}{2} (q_t - y)^2 \frac{\partial^2}{\partial y^2} \Lambda(t, y) = \sigma (q_t - y)^2 \frac{\partial^2}{\partial y^2} g(t, y)$$
Reduced Basis Methods and POD

Problem
Many solutions of the PDE are necessary for the Calculation of the Hedge weights in the simulation ($N \cdot 63$), each with different parameters.

$$\Longrightarrow$$ Approximate the solution with a reduced model.

Instead of the classical hat basis (FE)

try to use a reduced basis consisting of “empirical” eigenfunctions.
Reduced Basis for the Black-Scholes Model

Store snapshots of the solution $g(t_i, y_j; \theta)^{M,N}_{i,j=1}$ of the PDE in a matrix $Z = (g(t_i, y_j; \theta^{\ell}_{\ell=1})^{M,N,\ell}_{i,j,\ell=1}$ and calculate via the SVD the reduced basis.

Example: Calculation of $g(t, y)$ with $N_y = 801, M = 400$

Offline: Calculate the reduced basis.
Online: Use $\mathcal{N} \ll N$ degrees of freedom for the actual computation.
Reduced Basis – Efficiency in Black-Scholes Model

_FEM_-calculation with 801 basis functions: 35 seconds

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Reduced Basis – Efficiency in Black-Scholes Model

**FEM-calculation with 801 basis functions: 35 seconds**

**Computation of the POD-basis: 39 seconds**

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### Reduced Basis – Efficiency in Black-Scholes Model

**FEM**-calculation with 801 basis functions: 35 seconds  
Computation of the **POD**-basis: 39 seconds  
**POD**-calculation with 15 basis functions: 1.2 seconds

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Reduced Basis for the Heston Model

Example: Calculate $g(t, y, v)$ with $N_y = 61, N_v = 41, M = 625$
Reduced Basis – Efficiency in Heston Model

**FEM-calculation with $81 \times 61$ basis functions: 123 seconds**

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## Reduced Basis – Efficiency in Heston Model

**FEM-calculation with $81 \times 61$ basis functions:** 123 seconds  
**Computation of the POD-basis:** 132 seconds

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Reduced Basis – Efficiency in Heston Model

**FEM-calculation with** $81 \times 61$ **basis functions:** 123 seconds  
**Computation of the POD-basis:** 132 seconds  
**POD-calculation with** 42 **basis functions:** 1.7 seconds

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Approach

Scenario Generation

1. Choose a “true model” as well as its $\mathbb{P}$-and $\mathbb{Q}$ parameters.

2. Generate $N = 50000$ trajectories each with 63 days under $\mathbb{P}$ with daily observation of $S_t$ and calculate daily $\mathbb{Q}$-prices of $C^{E,1}, \ldots, C^{E,15}$. 
Approach

Scenario Generation

1. Choose a “true model” as well as its $\mathbb{P}$-and $\mathbb{Q}$ parameters.
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Hedge Calculation and Evaluation

1. Choose the maturity of $C^A$ as $\{21, 126, 189\}$ days.
2. Choose the hedge model.
3. For each trajectory the hedge model gets calibrated to $S_t$ and (a subset of) $C^{E,1}, \ldots, C^{E,15}$ on a daily basis.
4. Calculate the hedge portfolio.
5. At the end of the path the hedging error is evaluated.
6. Build up the hedging error distribution.
Hedging under Model Uncertainty

Considered Models

Reduced Model and POD

Results
3FM, Local Calibration, Delta-Hedge

BS in 3FM(21d)

BS in 3FM(126d)

BS in 3FM(189d)

SABR in 3FM(21d)

SABR in 3FM(126d)

SABR in 3FM(189d)
3FM, Local Calibration, Delta- and Vega-Hedge

- BS(vega) in 3FM(21d)
- BS(vega) in 3FM(126d)
- BS(vega) in 3FM(189d)

- SABR(vega) in 3FM(21d)
- SABR(vega) in 3FM(126d)
- SABR(vega) in 3FM(189d)
SVJJ, Local Calibration, Delta-Hedge
SVJJ, Local Calibration, Delta- and Vega-Hedge

BS(vega) in SVJJ(21d)

BS(vega) in SVJJ(126d)

BS(vega) in SVJJ(189d)

HEST(vega) in SVJJ(21d)

HEST(vega) in SVJJ(126d)

HEST(vega) in SVJJ(189d)
Hedging under Model Uncertainty

CGMYe, Local Calibration, Delta-Hedge

SABR in CGMYe(21d)

SABR in CGMYe(126d)

SABR in CGMYe(189d)

HEST in CGMYe(21d)

HEST in CGMYe(126d)

HEST in CGMYe(189d)
CGMYe, Local Calibration, Delta- and Vega-Hedge

SABR(vega) in CGMYe(21d)

SABR(vega) in CGMYe(126d)

SABR(vega) in CGMYe(189d)

HEST(vega) in CGMYe(21d)

HEST(vega) in CGMYe(126d)

HEST(vega) in CGMYe(189d)
3FM, Global Calibration, Delta-Hedge
3FM, Global Calibration, Delta- and Vega-Hedge

Hedging error

Adj. rel. Frequency

-1.0 -0.5 0.0 0.5

0 5 15

BS(vega) in 3FM(21d)

BS(vega) in 3FM(126d)

BS(vega) in 3FM(189d)

HEST(vega) in 3FM(21d)

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SVJJ, Global Calibration, Delta-Hedge
SVJJ, Global Calibration, Delta- and Vega-Hedge
Wrap up

- Hedging under model uncertainty
- POD for parameter dependent (parabolic) PDEs
- Analysis of the hedging error distribution
- Simple models (BS) seem to be preferable in unknown markets
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- POD for parameter dependent (parabolic) PDEs
- Analysis of the hedging error distribution
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Thank you for your attention