A general optimal stopping game with applications in finance

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(joint work with S. P. Yung/HKU and Phillip Yam/PolyU)

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Motivations: game call options and callable stock loans

Formulation of a general optimal stopping game

Solutions: perpetual case

Solutions: finite time horizon
Assuming a Black–Scholes market:

- A risk free bond $B$ with a constant riskless interest rate $r$,
  \[ dB_t = rB_t \, dt, \]
  where $dB_t$ is the infinitesimal change in the bond price.

- A stock with price process $S$, which under the risk neutral measure is governed by
  \[ dS_t = (r - d) S_t \, dt + \kappa S_t \, dW_t, \]
  where the interest rate $r \geq 0$, the dividend $d \geq 0$ and the volatility $\kappa > 0$ are constants and $W$ is a one-dimensional Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, Q)$ with $W_0 = 0$ almost surely.
At time 0 the option holder pays a premium to the option writer and at any time $t$ (before maturity) both the holder and the writer have the right to exercise the option. If the holder exercises the option at time $t$, he would claim the amount

$$Y_t = (S_t - K)^+$$

with strike price $K$. If the option writer exercises (or cancells) at time $t$, he is obliged to pay the holder the amount

$$Z_t = (S_t - K)^+ + \delta \geq Y_t$$

with a penalty $\delta$. 
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What is the no arbitrage initial price for game call options?
Initiated by Dynkin (1969) and later reformulated by Neveu (1975) to a more general set up.

Game option by Kifer (2000).

Kyprianou (2004): Perpetual game put options on stock without dividend payment:
- When penalty is large: option writer should never exercise (cancel) the contract;
- When penalty is small: exercising region for option writer is $\{K\}$.


Callable stock loans

- Stock loans: a loan in which the borrower (client), who owns one share of a stock, obtains a loan from the lender (bank) with the share as collateral.
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- At time 0, the client borrows amount $L$ from the bank with one share of the stock as collateral. The bank charges amount $m$ and a loan rate $\gamma$ for providing the service. So the client pays $S_0 + m - L$ at initial time.
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  - paying the lender the principal amount and the loan interest, which is equal to $L e^{\gamma t}$ and hence redeeming his share of stock, or
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- The payoff of the client is $Y_t = (S_t - L e^{\gamma t})^+$ when he terminate the contract at time $t$. 

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Callable stock loans

Callable stock loan: a stock loan with the additional feature that the lender can call back the loan at any time $t$ (before maturity).

- $\pi$: the portion of loan the bank requires the client to pay at the time of call.
- The bank pays the client the amount $Z_t = (S_t \pi L) + Y_t$. 
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The bank pays the client the amount $Z_t = (S_t \pi L \gamma_t) + Y_t$. Products with similar structure are traded on the financial markets under the name "callable repo."
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- The rational value of $L$ and $m$ should be such that the initial value of the callable stock loan is $(S_0 - L + m)$. 
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- initial value of callable stock loans: smallest initial capital for the lender of the loan to superhedge his position.
- The rational value of $L$ and $m$ should be such that the initial value of the callable stock loan is $(S_0 - L + m)$.
- What is the rationale values of $L$ and $m$?
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Literature review: Stock loans

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- Perpetual stock loans under Jump risk (Cai.N (2009))
A perpetual optimal stopping game

Let $X$ be a process under the risk neutral measure $Q_x$ satisfying

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$$dX_t = (\rho - d) X_t dt + \kappa X_t dW_t, \ X_0 = x.$$  

Then the infinitesimal generator of the process $(e^{-\rho t} X_t)_{0 \leq t < \infty}$ is given by

$$A \triangleq \frac{\kappa^2}{2} x^2 \frac{d^2}{dx^2} + (\rho - d) x \frac{d}{dx} - \rho.$$  

(1)
Define

\[ g_1(x) \triangleq (x - q)^+ , \quad g_2(x) \triangleq \max(x - q + c, \theta c)^+ \]

and

\[ R_{s,t} \triangleq g_1(X_t) \mathbf{1}_{\{ t \leq s \}} + g_2(X_t) \mathbf{1}_{\{ s < t \}}. \]
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Problem

Find a function \( v \) and a pair of stopping times \((\sigma^*, \tau^*)\) such that the following holds

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v(x) = \sup_{\tau \geq 0} \inf_{\sigma \geq 0} \mathbb{E}_x \left[ e^{-\rho \sigma \wedge \tau} R_{\sigma, \tau} \right]
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\]

\(\theta = 1\): game call options; \(\theta = 0\): callable stock loans.
Solution to the perpetual case:

Let $\lambda_1 > \lambda_2$ to be the roots of the quadratic equations

$$\frac{\kappa^2}{2} \lambda^2 + \left( \rho - d - \frac{\kappa^2}{2} \right) \lambda - \rho = 0.$$ 

and define

$$\lambda^* \triangleq \begin{cases} 
1 & \text{if } d = 0 \text{ and } \rho \geq -\frac{\kappa^2}{2} \\
\frac{1}{(\lambda_1 - 1)^{\lambda_1 - 1}} & \frac{\lambda^{\lambda_1}}{\lambda_1^{\lambda_1}} < 1 \text{ if } d > 0, \text{ or } d = 0 \text{ and } \rho < -\frac{\kappa^2}{2}.
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- When \( \theta c \geq \lambda^* q^{1-\lambda_1} (q - c + \theta c)^{\lambda_1} \)
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- The explicit form of $v$ in this case was given in Xia and Zhou (2007).
Solution to the perpetual case:

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  - Exercising region for option writer:
    
    $$D_2 = \{ x : v(x) = g_2(x) \} ;$$
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    \]
  - Exercising region for option holder:
    \[
    D_1 = \left\{ x : v(x) = g_1(x) \right\}.
    \]
Solution to the perpetual case:

- Take $\theta = 0$ as an example (the case of callable stock loan):
  
  $$g_1(x) = (x - q)^+, \ g_2(x) = (x - q + c)^+.$$

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- An observation:
  
  \[ g_1(x) = g_2(x) = 0 \text{ for } x \leq q - c. \]

  This implies
  
  \[ \nu(x) = 0 \text{ for } x \leq q - c \]

  and hence
  
  \[ (0, q - c] \subset D_2 \cap D_1. \]
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  and hence
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- This reflects the fact that $q - c$ is the amount of the loan that the bank can at least get.
Solution to the perpetual case:

- **Case 1:** $\rho \geq 0$ and $d = 0$.

  $$v(x) = g_2(x),$$

  and

  $$D_2 = (0, \infty), D_1 = (0, q - c]$$
Solution to the perpetual case:

- **Case 1:** \( \rho \geq 0 \) and \( d = 0 \).

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D_2 = (0, \infty), \ D_1 = (0, q - c]
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- In this case,

\[
\mathcal{A}g_i \geq 0 \text{ for } i = 1, 2.
\]

The bank exercises immediately; while the client don't exercise (as long as \( X_t > q - c \)): 
Solution to the perpetual case:

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Intuitively the client should stop as long as $X$ is large enough.

For the bank, when (i) $\rho < 0$, or (ii) $\rho = 0$ and $d > 0$, or (iii) $\rho > 0$ and $d \geq \rho > 0$,

$$Ag_2 = -dx + r(q - c) \leq 0 \text{ for } x > q - c.$$  

The bank should wait as long as $X_s > q - c$, i.e.

$$D_2 = (0, q - c].$$
Solution to the perpetual case:

- In the case \( \rho \geq 0 \) and \( d = 0 \),

\[
D_2 = (0, \infty).
\]

- In cases (i), (ii) and (iii),

\[
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- In the remaining case: $\rho > 0$ and $\rho > d > 0$, we expect $D_2$ changes continuously in terms of model parameter.
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- A critical dividend $\rho > d^* > 0$:
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  - case (iv) $\rho > 0$ and $\rho > d \geq d^*$ : $D_2 = (0, q - c]$;
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    - case (iv) $\rho > 0$ and $\rho > d \geq d^*$: $D_2 = (0, q - c]$;
    - case (v) $\rho > 0$ and $d^* > d > 0$: $D_2 = (0, b_1]$ with $b_1 \uparrow \infty$ as $d \downarrow 0$. 
Solution to the perpetual case:

- Case 2. a): When

\[ \rho < 0 \text{ or } \rho = 0 \text{ and } d > 0 \text{ or } \rho > 0 \text{ and } d > 0, \]

\[ D_2 = (0, q_{cc}], \quad D_1 = (0, q_{cc}]. \]
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(ii) $\rho = 0$ and $d > 0$ or
(iii) $\rho > 0$ and $d \geq \rho > 0$ or
(iv) $\rho > 0$ and $\rho > d \geq d^*$,

$D_2 = (0, q - c), D_1 = (0, q - c) \cup [a_0, \infty)$
Solution to the perpetual case:

- Plotting the value function:

\[
D_2 = (0, q - c] = (0, 64].
\]

**Figure:** Figure 3: Graphical illustration of \( v \) with a market model \( \rho = 0.03, \kappa = 0.15, d = 0.025, q = 80 \) and \( c = 16 \). In this case \( D_2 = (0, q - c] = (0, 64] \).
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Figure: Figure 3: Graphical illustration of \( v \) with a market model \( \rho = 0.03 \), \( \kappa = 0.15 \), \( d = 0.025 \), \( q = 80 \) and \( c = 16 \). In this case \( D_2 = (0, q - c] = (0, 64] \).

- Smooth fit principle fails at the lower boundary \( q - c \).```
Solution to the perpetual case:

- \( a_0 \triangleq \alpha_0 q \) and \( \alpha_0 \) is the unique solution to either of the following equations.

  - for the case \( d > 0 \) or \( \rho \neq -\frac{\kappa^2}{2} \),
    \[
    (1 - \lambda_1) \alpha^{1-\lambda_2} + \lambda_1 \alpha^{-\lambda_2} = \left( \frac{q - c}{q} \right)^{\lambda_1-\lambda_2} \left[ (1 - \lambda_2) \alpha^{1-\lambda_1} + \lambda_2 \alpha^{-\lambda_1} \right];
    \]

  - for the case \( d = 0 \) and \( \rho = -\frac{\kappa^2}{2} \),
    \[
    \alpha - \ln \alpha + \ln \frac{q - c}{q} - 1 = 0.
    \]
Solution to the perpetual case:

- When $\rho > 0$, define

$$v_B(x) \triangleq \sup_{\tau} \mathbb{E}_x \left[ e^{-\rho \tau} g_1(X_{\tau}) \mathbb{1}_{\{\tau < \sigma_{q-c}\}} \right]$$

and

$$d^* \triangleq \inf \left\{ d > 0 : \frac{d}{dx} v_B((q - c) +) \leq 1 \right\},$$

where $\frac{d}{dx} u((q - c) +) = \lim_{x \downarrow (q-c+\theta c)} \frac{u(x)}{x-(q-c)}$. 
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Solution to the perpetual case:

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- $v_B$: the price of an American down-and-out call option with strike $q$ and barrier $q - c$.

- $d^*$: the smallest dividend such that the delta of the American down-and-out call at the barrier is smaller than unity.
Case 2. b): When $\rho > 0$ and $d^* > d > 0$, 

\[ D_2 = (0, b_1] \text{ and } D_1 = (0, q - c] \cup [a_1, \infty) . \]
Solution to the perpetual case:

- Plotting the value function:

Figure: Figure 4: Graphical illustration of $v$ with a market model $\rho = 0.03$, $\kappa = 0.15$, $d = 0.012$, $q = 80$ and $c = 16$. In this case $D_2 = (0, b_1] = (0, 92.77]$. 

- Smooth fit principle holds at the lower boundary $b_1$. 

Solution to the perpetual case:

\[(b_1, a_1) \triangleq (\beta_1 q, \alpha_1 q)\] and \((\beta_1, \alpha_1)\) is the unique pair of solutions to the system of equations

\[
\begin{align*}
(1 - \lambda_1) \alpha^{1-\lambda_2} + \lambda_1 \alpha^{-\lambda_2} + (\lambda_1 - \lambda_2) \left( \beta^{1-\lambda_2} - \frac{q-c}{q} \beta^{-\lambda_2} \right) \\
= \beta^{\lambda_1-\lambda_2} \left( (1 - \lambda_2) \alpha^{1-\lambda_1} + \lambda_2 \alpha^{-\lambda_1} \right), \\
(1 - \lambda_1) \beta^{1-\lambda_2} + \lambda_1 \frac{q-c}{q} \beta^{-\lambda_2} + (\lambda_1 - \lambda_2) \left( \alpha^{1-\lambda_2} - \alpha^{-\lambda_2} \right) \\
= \alpha^{\lambda_1-\lambda_2} \left( (1 - \lambda_2) \beta^{1-\lambda_1} + \lambda_2 \frac{q-c}{q} \beta^{-\lambda_1} \right).
\]
Rational value of $L$ and $m$: consider the case 2.b) $r - \gamma > 0$ and $d < d^*$:
Rational value of $L$ and $m$: consider the case 2.b) $r - \gamma > 0$ and $d < d^*$:

$X_0 \geq a_1$, i.e. $\frac{L}{X_0} \leq \frac{L}{a_1} = \frac{1}{\alpha_1}$, the loan-to-value is too small for the client, and the optimal time to redeem the stock is $\tau_{a_1} = 0$, thus there is actually no physical exchange between the bank and the client.
Rational value of $L$ and $m$: consider the case 2.b) $r - \gamma > 0$ and $d < d^*$:

- $X_0 \geq a_1$, i.e. $\frac{L}{X_0} \leq \frac{L}{a_1} = \frac{1}{\alpha_1}$, the loan-to-value is too small for the client, and the optimal time to redeem the stock is $\tau_{a_1} = 0$, thus there is actually no physical exchange between the bank and the client.

- $X_0 \leq b_1$, i.e. $\frac{L}{X_0} \geq \frac{L}{b_1} = \frac{1}{\beta_1}$, the loan-to-value is too large for the bank, and the optimal call time is $\sigma_{b_1} = 0$, which also suggests that there is no exchange between the two parties again.
Rational value of $L$ and $m$: consider the case 2.b) $r - \gamma > 0$ and $d < d^*$:

- $X_0 \geq a_1$, i.e. $\frac{L}{X_0} \leq \frac{L}{a_1} = \frac{1}{\alpha_1}$, the loan-to-value is too small for the client, and the optimal time to redeem the stock is $\tau_{a_1} = 0$, thus there is actually no physical exchange between the bank and the client.

- $X_0 \leq b_1$, i.e. $\frac{L}{X_0} \geq \frac{L}{b_1} = \frac{1}{\beta_1}$, the loan-to-value is too large for the bank, and the optimal call time is $\sigma_{b_1} = 0$, which also suggests that there is no exchange between the two parties again.

- $X_0 \in (b_1, a_1)$, both parties are willing to carry out the business and the fair fee charged is $m = v(X_0) - X_0 + L$, i.e. the loan is marketable if the loan-to-value ratio lies in $\left(\frac{L}{a_1}, \frac{L}{b_1}\right) = \left(\frac{1}{\alpha_1}, \frac{1}{\beta_1}\right)$. 
optimal stopping game in finite time horizon

Problem

Find a function \( v \) and a pair of stopping times \((\sigma^*, \tau^*)\) such that the following holds

\[
v(t, x) = \sup_{\tau \leq T-t} \inf_{\sigma \leq T-t} \mathbb{E}_{t,x} \left[ e^{-\rho \sigma \wedge \tau} R_{\sigma, \tau} \right]
\]

\[
= \inf_{\sigma \leq T-t} \sup_{\tau \leq T-t} \mathbb{E}_{t,x} \left[ e^{-\rho \sigma \wedge \tau} R_{\sigma, \tau} \right] = \mathbb{E}_{t,x} \left[ e^{-\rho \sigma^* \wedge \tau^*} R_{\sigma^*, \tau^*} \right].
\]
When \( \theta c \geq \lambda^* q^{1-\lambda_1} (q - c + \theta c)^{\lambda_1} \), the above problem becomes an optimal stopping problem, it is never for the option writer to stop.
Solution to the finite maturity problem

When $\theta c \geq \lambda^* q^{1-\lambda_1} (q - c + \theta c)^{\lambda_1}$, the above problem becomes an optimal stopping problem, it is never for the option writer to stop.

When $\theta c < \lambda^* q^{1-\lambda_1} (q - c + \theta c)^{\lambda_1},$
Solution to the finite maturity problem

- When \( \theta c \geq \lambda^* q^{1-\lambda_1} (q - c + \theta c)^{\lambda_1} \), the above problem becomes an optimal stopping problem, it is never for the option writer to stop.
- When \( \theta c < \lambda^* q^{1-\lambda_1} (q - c + \theta c)^{\lambda_1} \),
  - Case 1: \( \rho \geq 0 \) and \( d = 0 \).

\[
v(t, x) = \begin{cases} 
\frac{\theta c}{q-c+c} x & \text{if } x \leq q - c + \theta c \\
 x - q + c & \text{if } x > q - c + \theta c 
\end{cases}
\]
Solution to the finite maturity problem

- When $\theta c \geq \lambda^* q^{1-\lambda_1} (q - c + \theta c)^{\lambda_1}$, the above problem becomes an optimal stopping problem, it is never for the option writer to stop.

- When $\theta c < \lambda^* q^{1-\lambda_1} (q - c + \theta c)^{\lambda_1}$,
  
  Case 1: $\rho \geq 0$ and $d = 0$.

  $$v(t, x) = \begin{cases} \frac{\theta c}{q-c+\theta c} x & \text{if } x \leq q - c + \theta c \\ x - q + c & \text{if } x > q - c + \theta c \end{cases}.$$ 

- Case 2: $\rho < 0$ or $d > 0$. 
Solution to the finite maturity problem

Take $\theta = 0$ as an example

- In case 2.a)

Plotting of optimal stopping boundaries in a market model with $\rho = -0.03$, $\kappa = 0.15$, $d = 0$, $q = 100$, $c = 16$ and $T = 20$. 
Solution to the finite maturity problem

Consider $\theta = 0$ and case 2.a)

- $b^*(t) \equiv \pi L$ and $a^*(t) = a_0(t)$ is the unique solution to the integral equation

$$I(t, a(t)) = a(t) - L + \int_0^{T-t} K_1(t, a(t), s, a(t + s)) \, ds$$

with terminal condition $a(T) = L$ if $\tilde{\rho} < 0$ or $d \geq \tilde{\rho}$ and $a(T) = \frac{\tilde{\rho}}{d} L$ otherwise, where

$$I(t, x) = \mathbb{E}_{t,x} \left[ e^{-\tilde{\rho}(T-t)} (X_T - L) + 1_{\{\tau_{\pi L} > T-t\}} \right],$$

$$K_1(t, x, s, y) = \mathbb{E}_{t,x} \left[ e^{-\tilde{\rho}s} (-dX_{t+s} + \tilde{\rho}L) 1_{\{X_{t+s} > y\}} 1_{\{\tau_{\pi L} > s\}} \right],$$

for $t \in [0, T]$ and $s \in [0, T-t]$. 
Solution to the finite maturity problem

- In case 2. b).

Plotting of optimal stopping boundaries in a market model with $\rho = 0.02$, $d = 0.01$, $\kappa = 0.15$, $d = 0.014$, $c = 16$, $q = 100$ and $T = 20$.

- The lower boundary is time dependent for $t < t^* = 13.3$. 
Consider $\theta = 0$ and case 2.b)

- Define

$$

\nu_B(s, x) \triangleq \sup_{\tau \in \mathcal{T}_{0,u}} \mathbb{E}_x \left[ e^{-\rho \tau} (X_\tau - q)^+ 1_{\{\tau \leq \tau_{q-c}\}} \right],

$$

(2)

and

$$

s^* = \sup \left\{ u > 0 : \frac{d}{dx} \nu_B(s, (q - c)^+) \leq 1 \right\}.

$$

(3)
Consider $\theta = 0$ and case 2.b)

- Define

$$v_B(s, x) \triangleq \sup_{\tau \in \mathcal{I}_{0,u}} \mathbb{E}_x \left[ e^{-\rho \tau} (X_\tau - q)^+ 1_{\{\tau \leq \tau_{q-c}\}} \right], \quad (2)$$

and

$$s^* = \sup \left\{ u > 0 : \frac{d}{dx} v_B(s, (q - c) +) \leq 1 \right\}. \quad (3)$$

- $v_B(s, x)$ is the price of a American down and out call option with time to maturity $s$, strike $q$ and barrier $q - c$. 
Consider $\theta = 0$ and case 2.b)

- Define

$$\nu_B (s, x) \overset{\Delta}{=} \sup_{\tau \in T_{0,u}} \mathbb{E}_X \left[ e^{-\rho \tau} (X_\tau - q)^+ \mathbb{1}_{\{\tau \leq \tau_{q-c}\}} \right], \quad (2)$$

and

$$s^* = \sup \left\{ u > 0 : \frac{d}{dx} \nu_B (s, (q - c) +) \leq 1 \right\}. \quad (3)$$

- $\nu_B (s, x)$ is the price of a American down and out call option with time to maturity $s$, strike $q$ and barrier $q - c$.
- $s^*$ is well-defined and $0 < s^* < \infty$. Define $t^* = (T - s^*) \lor 0$. 

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For $t > t^*$, $b^*(t) \equiv q - c$ and $a^*(t) = a_0(t)$ as in the case 2.a).
For $t > t^*$, $b^*(t) \equiv q - c$ and $a^*(t) = a_0(t)$ as in the case 2.a).

For $t \leq t^*$, $b^*(t) = b_1(t) > q - c$ and $a^*(t) = \alpha_1(t)$, where $(b_1, a_1)$ is the unique solution to the system of equations:

$$J(t, a(t)) = a(t) - q + \int_0^{T-t} K_1(t, a(t), s, a(t+s)) \, ds$$
$$+ \int_0^{T-t} K_2(t, a(t), s, b(t+s)) \, ds$$

$$J(t, b(t)) = b(t) - (q - \delta) + \int_0^{T-t} K_1(t, b(t), s, a(t+s)) \, ds$$
$$+ \int_0^{T-t} K_2(t, b(t), s, b(t+s)) \, ds$$

with terminal condition $a_1(t^*) = a_0(t^*)$ and $b_1(t^*) = q - c$. 
The function $J$ and $K_2$ are defined as

$$J(t, x) = \mathbb{E}_{t,x} \left[ e^{-\tilde{r}(t^*-t)} u\left(t^*-t, X_{t^*}\right) \mathbf{1}_{\{\tau_{\pi L}>t^*-t\}} \right],$$

$$K_2(t, x, s, y) = \mathbb{E}_{t,x} \left[ e^{-\tilde{r}s} \left( -dX_{t+s} + \tilde{r} \pi L \right) \mathbf{1}_{X_{t+s}<y} \mathbf{1}_{\{\tau_{\pi L}>s\}} \right],$$

for $t \in [0, t^*]$ and $s \in [0, t^* - t]$. 