Tangent Lévy Models

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Problem Formulation

Consider a *liquid market* consisting of an underlying price process $(S_t)_{t \geq 0}$ and prices of European Call options of all strikes $K$ and maturities $T$:

$$ \left( \{C_t(T, K)\}_{T,K>0} \right)_{t \geq 0} $$

Want to describe a large class of *market models*: arbitrage-free stochastic models (say, given by SDE's) for time-evolution of the market, $S$ and $\{C(T, K)\}_{T,K>0}$, such that

1. one can start the model from "almost" any *initial condition*, which is the set of currently observed market prices;
2. one can prescribe "almost" any *dynamics* for the model provided it doesn't contradict the no-arbitrage property.
Motivation

- Many Call Options have become liquid ⇒ need for financial models consistent with the observed option prices.

- Common *stochastic volatility* models (BS, Hull-White, Heston, etc.) are unable to reproduce the observed call prices of all strikes and maturities (fit the *implied volatility* surface).

  *Local volatility* models can fit option prices better.

- However, the above models have to be *recalibrated* to fit option prices at different times ⇒ they cannot be used to describe *time evolution* of call price surface.
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- However, the above models have to be recalibrated to fit option prices at different times ⇒ they cannot be used to describe time evolution of call price surface.
Preceding Results

- E. Derman, I. Kani (1997): idea of ”dynamic local volatility” for continuum of options.


Direct approach

- First, need a reasonable notion of "price" in the model: let's agree that pricing is linear, that is, **prices of all contingent claims are given by discounted conditional expectations of their payoffs under some measure** (assume discount rate is one).

- It seems natural to model "observables" directly under pricing measure: choose a driving *Brownian motion* $B$ and a *Poisson random measure* $N$ (which represent the background stochastic factors) and prescribe dynamics of (infinite-dimensional) *process of option prices* through its *semimartingale characteristics*

$$dC_t = \alpha_t dt + \beta_t \cdot dB_t + \int \gamma_t(x) [N(dx, dt) - \nu(dx, dt)]$$
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$$dC_t = \alpha_t dt + \beta_t \cdot dB_t + \int \gamma_t(x) [N(dx, dt) - \nu(dx, dt)]$$
Need to make sure these dynamics, indeed, produce option prices: each resulting $C_t(T, K)$ should coincide with corresponding conditional expectation.

\[ F(\alpha_t, \beta_t, \gamma_t) = 0, \]

where $F$ is known explicitly, and the above equation can be solved for some of the arguments.
Consistency conditions

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\[ \Downarrow \]

Consistency conditions on \{\(\alpha, \beta, \gamma\}\}

These conditions should be explicit! A perfect example is

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where \(F\) is known explicitly, and the above equation can be solved for some of the arguments.
Direct approach: difficulties

Turns out, the above direct approach (prescribing $dC_t$ directly) results in way too complicated consistency conditions...

Why does it happen?

- Recall that the definition of call prices as expectations implies certain "static no-arbitrage properties": $C_t(T, K)$ has to be nonnegative, convex in $K$, converge to payoff, etc. These properties have to be preserved by the dynamics, which is reflected in the consistency conditions - hence the complexity.

- Static no-arbitrage conditions define a manifold in space of functions of two variables. Therefore, the "consistent" set of parameters can only be of the form

\[ \alpha(C_t, t, \omega), \beta(C_t, t, \omega), \gamma(C_t, t, \omega) \]

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- Need to analyze resulting SDE in an "infinite-dimensional manifold"...
Let's *linearize* this manifold: find a one-to-one mapping of the set of feasible Call price surfaces (or its large enough subset) into some *open set in a linear space*. And consider dynamics in this linear space instead.

In general, **code-book** for a given set of derivatives is a one-to-one mapping defined on a family of their feasible price sets. Examples of code-books include:

- *Yield curve* for Treasury Bonds market.
- *Implied correlation* for CDO tranches.
- *Implied volatility* for Call options

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General methodology

**Code-books**

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Local Volatility as a code-book

- **B. Dupire (1994)** deduced that, if

  \[ d\tilde{S}_T = \tilde{S}_T a(T, \tilde{S}_T) dW_T, \quad \tilde{S}_0 = S_t, \]  

  then

  \[ a^2(T, K) := \frac{2}{K^2} \frac{\partial}{\partial T} C(T, K) \frac{\partial^2}{\partial K^2} C(T, K) \]  

  (1)

  We can use (2) to recover Local Volatility "a" from market prices of Call options, and

  we can use (1) to generate a (feasible!) Call price surface from a given Local Vol (and current level of underlying \( S_t \)).

  Only some regularity and nonnegativity is required from surface \( a(., .) \)!
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Other code-books

- When can we use Local Vol as a (static) code-book for Call prices?
  
  I. Gyongy: it is possible if underlying follows regular enough Ito process.

- Can we develop a general approach to construction of code-books?

- Local Volatility code-book can be interpreted as follows: we choose a model from the class of diffusion models, such that it produces the correct (market-given) call prices, and the corresponding Local Vol gives the code-book value.
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Consider a class of ”simple” financial models for the underlying, parameterized by $\theta \in \Theta$

$$\mathcal{M} = \{ M(\theta) \}_{\theta \in \Theta}$$

For example, $\mathcal{M}$ can be a class of diffusion models parameterized by Local Vol and initial value: $\theta = \left( a(.,.), \tilde{S}_0 \right)$.

Each model $M(\theta)$ produces Call prices $C^\theta(T, K)$. If the mapping $\theta \mapsto C^\theta$ is invertible, we obtain a code-book associated with $\mathcal{M}$.

Of course, $\Theta$ needs to be an open set in a linear space - but usually this can be achieved.

We have rediscovered calibration, but with a proper meaning now!
Tangent models

- Construct market model by prescribing time-evolution of $\theta_t$, and obtain $C_t$ as an inverse of the code-book transform.

- Recall that "feasibility" of call prices means there is a "true" (but unknown) martingale model for underlying process $S$ in the background.

- If at time $t$ there exists $\theta_t \in \Theta$, such that $C^{\theta_t}$ coincides with "true" Call price surface $C_t$, we say that the "true" model admits a tangent model from class $\mathcal{M}$ at time $t$.

- In the above notation, process $(\theta_t)_{t \geq 0}$ is consistent with a "true" model for $S$ if $M(\theta_t)$ is tangent to this "true" model at any time $t$. Note the analogy with tangent vector field in differential geometry.
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Note the analogy with tangent vector field in differential geometry.
Consider a model $M(\kappa, s)$, given by

- Exponential of a **pure jump additive (time-inhomogeneous Lévy)** process

$$
\tilde{S}_T = s + \int_t^T \int_\mathbb{R} \tilde{S}_u (e^x - 1) \left[ N(dx, du) - \nu(dx, du) \right],
$$

where $N(dx, du)$ is a **Poisson random measure** associated with jumps of $\log(\tilde{S})$, given by its compensator

$$
\nu(dx, du) = \kappa(u, x) dx du
$$

- equipped with its natural filtration.

Thus, we obtain the set of ”simple” models $\mathcal{M} = \{M(\kappa, s)\}$, with $\kappa$ changing in a space of (time-dependent) Lévy densities.
Lévy density as a code-book

- Notice that $C^{\kappa,s}(T, e^x)$ satisfies a PIDE analogous to the Dupire’s equation.

- Introduce $\Delta^{\kappa,s}(T, x) = -\partial_x C^{\kappa,s}(T, e^x)$, and deduce an initial-value problem for $\Delta^{\kappa,s}$ from the PIDE for call prices.

- Take Fourier transform in "$x" to obtain $\hat{\Delta}^{\kappa,s}(T, \xi)$. The initial-value problem in Fourier domain can be solved in closed form, which gives us an explicit expression for $\hat{\Delta}^{\kappa,s}$ in terms of $\kappa$ and $s$. This expression can be inverted to obtain $\kappa$ from $\hat{\Delta}^{\kappa,s}$ and $s$.

- Thus, given $s (= S_t)$, we have a bijection: $C^{\kappa,s} \leftrightarrow \Delta^{\kappa,s} \leftrightarrow \hat{\Delta}^{\kappa,s} \leftrightarrow \kappa$. 
Tangent Lévy Models

We say that \((S_t)_{t \in [0, \bar{T}]}\) and \((\kappa_t)_{t \in [0, \bar{T}]}\) form a tangent Lévy model if the following holds under the pricing measure:

1. \(C^{\kappa_t, S_t} = C_t\) at each \(t\).
2. Process \(S\) is a martingale, and \(\kappa_t \geq 0\).
3. \(S\) and \(\kappa\) evolve according to

\[
\begin{align*}
S_t &= S_0 + \int_0^t \int_\mathbb{R} S_u - (\exp(\gamma(\omega, u, x)) - 1)(N(dx, du) - \rho(x)dxdu), \\
\kappa_t &= \kappa_0 + \int_0^t \alpha_u du + \sum_{n=1}^m \int_0^t \beta_u^n dB_u^n,
\end{align*}
\]

where

- \(B = (B^1, \ldots, B^m)\) is a \(m\)-dimensional Brownian motion,
- \(N\) is a Poisson random measure with compensator \(\rho(x)dxdu\),
- \(\gamma(\omega, t, x)\) is a predictable random function,
- processes \(\alpha\) and \(\{\beta^n\}_{n=1}^m\) take values in a corresponding function space.
Consistency conditions

Given that 2 and 3 hold, 1 is equivalent to the following pair of conditions:

1. **Drift restriction:**

\[
\alpha_t(T, x) = Q(\beta_t; T, x) := \\
- e^{-x} \sum_{n=1}^{m} \int_{\mathbb{R}} \int_{t}^{T} \partial_{y}^{2} \psi_{\beta_t}^{n}(u; y) \, du \left[ \psi_{\beta_t}^{n}(T; x - y) \right] \\
- (1 - y \partial_{x}) \psi_{\beta_t}^{n}(T; x) \right] - \int_{t}^{T} \psi_{\beta_t}^{n}(u; y) \, du \psi_{\beta_t}^{n}(T; x - y) \, dy
\]

2. **Compensator specification:**

\[
\kappa_t(t, x) \, dx \, dt = (\rho(x) \, dx \, dt) \circ \gamma^{-1}(t, .)
\]

where \( \psi_{\beta_t}^{n}(T, x) = - e^{x} \int_{x}^{\text{sign}(x)\infty} \beta_t^{n}(T, y) \, dy \)
Existence of Tangent Lévy Models

**Specifications**

- Choose \( \rho(x) := e^{-\lambda|x| (|x|^{-1-2\delta} \vee 1)} \), with some fixed \( \lambda > 1 \) and \( \delta \in (0, 1) \).

- Consider \( \kappa \) of the form: \( \kappa(T, x) = \rho(x) \tilde{\kappa}(T, x) \), where \( \tilde{\kappa} \) is an element of the space of continuous functions, equipped with usual "sup" norm.

- Then \( \tilde{\alpha}_t = \alpha_t / \rho \) and \( \tilde{\beta}_t = \beta_t / \rho \), and we have

\[
d\tilde{\kappa}_t = \tilde{\alpha}_t \, dt + \tilde{\beta}_t \cdot dB_t,
\]

stopped at \( \tau_0 = \inf \left\{ t \geq 0 : \inf_{T \in [t, \bar{T}], x \in \mathbb{R}} \tilde{\kappa}_t(T, x) \leq 0 \right\} \).

- Then, \( \kappa_t := \rho \tilde{\kappa}_{t \wedge \tau_0} \) is nonnegative and changes on an open set in a linear space!

- There exists a (tractable) specification \( \gamma(t, x) := \Gamma(\tilde{\kappa}_t; x) \) which fulfills the "compensator specification" automatically.
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Local existence

\[
\begin{cases}
S_t = S_0 + \int_0^t \int_{\mathbb{R}} S_u - (\exp(\Gamma(\tilde{\kappa}_u; x)) - 1)(N(dx, du) - \rho(x)dxdu) \\
\tilde{\kappa}_t = \tilde{\kappa}_0 + \int_0^{t \wedge \tau_0} Q(\rho \tilde{\beta}_u) du + \sum_{n=1}^m \int_0^{t \wedge \tau_0} \tilde{\beta}_n u dB_u^n
\end{cases}
\]

(3)

For any given Poisson random measure $N$, with compensator $\rho(x)dxdt$, any Brownian motion $B = (B^1, \ldots, B^m)$ independent of $N$, and any progressively measurable square integrable stochastic processes $\{\tilde{\beta}_n\}_{n=1}^m$ (with values in corresponding function space) independent of $N$, there exists a unique pair $(S_t, \tilde{\kappa}_t)_{t \in [0, \bar{T}]}$ of processes satisfying (3). **The pair $(S_t, \rho \tilde{\kappa}_t \wedge \tau_0)_{t \in [0, \bar{T}]}$ forms a tangent Lévy model.**
Example of a tangent Lévy model

- Choose $m = 1$, and $\tilde{\beta}_t(T, x) = \xi_t C(x)$, where $C(x)$ is some fixed function (satisfying some technical conditions), and

$$\xi_t = \xi(\tilde{\kappa}_t) = \frac{\sigma}{\epsilon} \left( \inf_{T \in [t, \bar{T}], x \in \mathbb{R}} \tilde{\kappa}_t(T, x) \wedge \epsilon \right)$$

- Then ”drift restriction” simplifies to

$$Q(\rho \tilde{\beta}_t; T, x) = -\frac{e^{-x}}{\rho(x)} \int_{\mathbb{R}} \int_t^T \partial_y \psi^{\rho \tilde{\beta}_t} (u, y) \, du \partial_x \psi^{\rho \tilde{\beta}_t} (T, x - y)$$

$$- \int_t^T \psi^{\rho \tilde{\beta}_t} (u, y) \, du \psi^{\rho \tilde{\beta}_t} (T, x - y) \, dy = \xi^2(\tilde{\kappa}_t)(T - t \wedge T) A(x)$$

and

$$\tilde{\kappa}_t(T, x) = \tilde{\kappa}_0(T, x) + (T - t \wedge T) A(x) \int_0^t \xi^2(\tilde{\kappa}_u) \, du + C(x) \int_0^t \xi^2(\tilde{\kappa}_u) \, dB_u$$
Conclusions

- We have described a general approach to constructing market models for Call options: find the right code-book by choosing a space of tangent models, prescribe time-evolution of the code-book value via its semimartingale characteristics and analyze consistency of resulting dynamics.

- This approach was illustrated by "Tangent Lévy Models" - a large class of market models, explicitly constructed and parameterized by $\tilde{\beta}$!

- Proposed market models allow one to start with observed call price surface and model explicitly its future values under the risk-neutral measure. For example, they provide a flexible framework for simulating the (arbitrage-free) evolution of implied volatility surface.
Further extensions

- One needs to consider $\tilde{\beta}_t = \tilde{\beta}(\tilde{\kappa}_t)$ and solve the resulting SDE for $\tilde{\kappa}_t$, as shown in the example, in order to ensure that $\tilde{\kappa}$ stays positive.

- There exists an extension of the Lévy-based code-book, the pair ("Lévy density", "instantaneous volatility"), which allows the "true" underlying to have a non-trivial continuous martingale component.
Estimated coefficients $C^1$ and $C^2$, as functions of $x = \log(K/S)$. 