Dynamic Consumption and Portfolio Choice with Ambiguity about Stochastic Volatility

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Outline:

- Concept: Ambiguity vs Risk
- Motivation
- Model
- Analytical Solution
- Simulation Results
- Conclusions

- Knight (1921): conceptual distinction between ambiguity and risk.
  - Risk: uncertainty that can be described by a single probability distribution. “known unknown”.
  - Ambiguity: uncertainty than cannot be described by a single probability distribution. “unknown unknown”.


Variation on Ellsberg (1961, QJE) 2-colour, 2-urn experiment:

<table>
<thead>
<tr>
<th>Ambiguous Urn – 100 Balls</th>
<th>Risky Urn – 100 Balls</th>
</tr>
</thead>
<tbody>
<tr>
<td>?? Red</td>
<td>50 Red</td>
</tr>
<tr>
<td>?? Blue</td>
<td>50 Blue</td>
</tr>
</tbody>
</table>

**Question:** Placed in a choice situation, which urn does the typical agent choose to draw a ball?

**Answer:** Strict preference for betting on the Risky urn. Why?
- The chance of winning (50% in this case) is “safe” and well understood.

**Implication from RR $\succ AR$ and RB $\succ AB$:**

- As $Pr(RR)=Pr(RB)=0.5$, then implied “subjective” probabilities are $Pr(AR) < 0.5$ and $Pr(AB) < 0.5$. Paradox!
- Standard Additive Probability can not represent Ellsberg evidence about agent’s behavior in such uncertain context.

- Mainstream Theory of Choice in Economics for the last 60 years:
  - (EU) Expected Utility Theory (von Neumann and Morgenstern, 1944):
    - Probabilities of the possible states of nature are known.
  - (SEU) Subjective Expected Utility Theory (Savage, 1954):
    - Probabilities are not necessarily known, but agents still behave as if they were maximizing an expected utility function, using their subjective probability beliefs.
  - Both EU and SEU ignore ambiguity, reducing all uncertainty to risk.

- Gradually, ambiguity is being incorporated in decision theory since 90’s: (i) further empirical evidence; (ii) theoretical developments (Multiple-Priors Approach and Robust Control).

- What is the impact on the dynamic consumption and portfolio choices from the ambiguity about the stochastic precision*?

- Is stochastic precision relevant to portfolio choice?

* Note: precision is the reciprocal of variance (volatility) of the risky asset’s return.
2.1 Motivation: Literature Review.

- Large literature on the portfolio choice problem without ambiguity considerations.

  - Few of those works explore the problem with stochastic precision.

- Few and recent literature focuses on portfolio choice with ambiguity aversion, but:

  - Ambiguity is about the expected (excess) return of the risky asset.

  - No explicit stochastic process for precision.
2.2 Motivation: Ambiguity about Expected Precision?

- This paper introduces Ambiguity aversion:
  - within a setting with an explicit process for the stochastic precision.
  - about the expected value of precision of the risky asset’s return.

- Why?
  - Precision: not observed by investors - intuitive reason to assume they may feel ambiguous on it.
  - Precision’s expected value: the most intuitive parameter to which investors pay attention.
  - Analytical tractability.

- Chacko and Viceira (2005) – base model:
  - For dynamic consumption and portfolio choice.
  - Instantaneous return of the risky asset given by:

\[
\frac{dS_t}{S_t} = \mu dt + \sqrt{\frac{1}{y_t}}dW_S
\]  

(1)

- Precision \( y_t \) follows a mean-reverting square-root process described by:

\[
dy_t = \kappa (\theta - y_t) dt + \sigma \sqrt{y_t}dW_y,
\]  

(2)

where:
- \( \mu \) - expected return of the risky asset
- \( \text{E}(y_t) = \theta \);
- \( W_s \) and \( W_y \) are standard Brownian Motions; assumed \( dW_ydW_S = \rho dt , \rho > 0 \).
Preferences are represented by the Stochastic Differential Utility function introduced by Duffie and Epstein (1992), with the utility process:

\[ J = E_t \left[ \int_t^\infty f(C_s, J_s) \, ds \right], \quad (3) \]

where:

- \( C_s \) - current consumption
- \( J_s \) - continuation utility on \( C \) at time \( t=s \)
- \( f(C_s, J_s) \) - normalized aggregator that generates \( J \). It is a function of, among others:
  - \( \gamma > 0 \) - coefficient of relative risk aversion
  - \( \psi > 0 \) - elasticity of intertemporal substitution of consumption
  - \( \beta > 0 \) - the rate of time preference.
3.1 Model: Our contribute.

Our contribution:

☐ Assume ambiguity about $E(y_t) = \theta$.

☐ Following Gilboa and Schmeidler (1989) Max-Min framework and applying the Saddle Point Theorem [Fan(1953), Sion(1958)]:

☐ Investors have a set of priors, the interval $[\underline{\theta}, \overline{\theta}]$, with $0 < \underline{\theta} \leq \theta \leq \overline{\theta}$

☐ Investors consider $\theta^* \in [\underline{\theta}, \overline{\theta}]$ such that it minimizes the maximized expected utility:

$$\theta^* = \arg\min_{\hat{\theta} \in [\underline{\theta}, \overline{\theta}]} J_{t_0}(\hat{\theta}).$$  \hspace{1cm} (4)
The dynamic consumption-portfolio problem with stochastic precision faced by the investor that is both $\theta$-ambiguity and risk averse can be written as:

$$
\min_{\tilde{\theta} \in [\theta, \bar{\theta}]} \left\{ \max_{\pi, C} E_{t_0} \left[ \int_{t_0}^{\infty} f(C_s, J_s) \, ds \right] \right\}
$$

subject to

$$
dX_t = [\pi_t (\mu - r) X_t + rX_t - C_t] \, dt + \pi_t \sqrt{\frac{1}{y_t}} X_t \, dW_S,
$$

$$
dy_t = \kappa (\tilde{\theta} - y_t) \, dt + \sigma \sqrt{y_t} \, dW_y.
$$

where:

- $X_t$ - wealth (with $X_{t_0} > 0$)
- $\pi_t$ - fraction of wealth invested in the risky asset.
For each $\hat{\theta}$, the maximization problem is a stochastic continuous-time optimal control problem with two state variables ($X_t$ and $y_t$) and two control variables ($C_t$ and $\pi_t$). The corresponding Bellman equation is:

$$0 = \max_{\pi,C} \left\{ f(C_s,J_s) + J_X(\pi_t(\mu - r)X_t + rX_t - C_t) + J_y(\hat{\theta} - y_t) + \frac{1}{2} J_{XX} \pi_t^2 \frac{1}{y_t}X_t^2 + \frac{1}{2} J_{yy} \sigma_t^2 y_t + J_{xy} \pi_t \sigma X_t \right\}.$$  

(6)

where $J_{(.)}$ are partial derivatives.

Chacko and Viceira (2005) found an exact solution when $\psi = 1$ and an approximate solution for $\psi \neq 1$. We study $\theta$ – ambiguity in both scenarios.
4.1 Problem Solution: Exact Solution.

When $\psi = 1$ the value function $J$ that solves (6), for any value of $\hat{\theta}$, is given by:

$$J(\hat{\theta}, X_t, y_t) = \exp \left\{ Ay_t + B(\hat{\theta}) \right\} \frac{X_t^{1-\gamma}}{1-\gamma},$$  (7)

where $A$ and $B$ are constants depending on parameters describing investors preferences and the investment opportunity set. Optimal consumption and portfolio rules are given by:

$$C_t = \beta X_t,$$  (8)

$$\pi_t = \frac{1}{\gamma} (\mu - r) y_t + \frac{\sigma \rho}{\gamma} A y_t.$$  (9)

From (8), optimal consumption choice does not depend on $y_t$. From (9) and considering $E(y_t) = \theta$, the mean optimal allocation in the risky asset is given by:

$$\pi_\theta = \frac{1}{\gamma} (\mu - r) \theta + \frac{\sigma \rho}{\gamma} A \theta.$$  (10)
4.1 Problem Solution: Exact Solution.

- What happens with the introduction of $\theta$ - ambiguity aversion?

- New $\theta$ value ($= \theta^*$) in accordance with (4):

$$\theta^* = \arg\min_{\hat{\theta} \in [\underline{\theta}, \overline{\theta}]} J_{t_0}(\hat{\theta}).$$

$\Rightarrow$ Proposition 1
4.1 Problem Solution: Exact Solution.

Comments on Proposition 1:
- Domain of the solution of the ambiguity problem depends on the relation between:
  - the level of relative risk aversion (γ)
  - characterization of the investment opportunity dynamics (ω).
- Under that domain, γ ≥ ω, precision is always good.

When ψ = 1 and γ ≥ ω, where ω = \( \frac{\sigma^2(\mu_r - \bar{\mu})^2 + 2\sigma(\mu_r - \bar{\mu})(\beta + \kappa)}{(\beta + \kappa)^2 + 2\sigma^2(\mu_r - \bar{\mu})(\beta + \kappa)} < 1 \), the solution of the ambiguity problem is:

\[ \theta^* = \frac{1}{\theta}. \]
4.1 Problem Solution: Exact Solution.

- Impact on Optimal Consumption and Portfolio rules?
  - None, as (8) and (9) do not depend on θ.

- Is θ – ambiguity aversion irrelevant?
  - No, if ambiguity averse investor observes the instantaneous precision but…can not adjust instantaneously his portfolio (e.g. transaction costs, human limitations):
    - expectation of future precision, and not instantaneous precision, drives investor’s portfolio decision.

  - mean allocation to the risky asset differs from (10).

  => Proposition 2
4.1 Problem Solution: Exact Solution.

Proposition 2 – Portfolio choice under “expectation-driven” scenario

When $\psi = 1$, $\gamma \geq \omega$, and the $\theta$-ambiguity averse investor considers the expected precision of the risky asset return instead of the instantaneous precision, the demand for the risky asset is:

$$\pi_\theta = \frac{1}{\gamma} (\mu - r) \theta + \frac{\sigma \rho}{\gamma} A \theta,$$  \hspace{1cm} (11)

which can be decomposed into three components:

- myopic demand $= \frac{1}{\gamma} (\mu - r) \theta$  \hspace{1cm} (12)
- intertemporal hedging demand $= \frac{\sigma \rho}{\gamma} A \theta$  \hspace{1cm} (13)
- ambiguity demand $= \left[ \frac{1}{\gamma} (\mu - r) + \frac{\sigma \rho}{\gamma} A \right] (\theta - \theta)$.  \hspace{1cm} (14)

☐ Comment on Proposition 2: New - introduction of the ambiguity demand component (14).
Proposition 3 – $\theta$ - ambiguity aversion impact on the demand for the risky asset (expectation-driven scenario):

(i) $\theta$-ambiguity aversion reduces the mean allocation to the risky asset;

(ii) Ambiguity demand (14) is always negative;

(iii) Intertemporal hedging demand is negative if $\gamma > 1$ and positive if $\omega \leq \gamma < 1$. 
In Chacko and Viceira (2005) it is found that the intertemporal hedging demand is empirically small:

- Calibration with long-run US data: monthly excess stock returns on the CRSP value-weighted portfolio over the T-Bill rate (January 1926 – December 2000)

- Conclusion: “risk dimension” of stochastic precision is not relevant for the portfolio decision.

Our question: What happens, under the expectation-driven scenario, if ambiguity on stochastic precision is considered?
5 Simulation: Guidelines.

The same calibration as in Chacko and Viceira (2005):

\[ \mu - r = 0.0811 \]
\[ \kappa = 0.3374 \]
\[ \theta = 27.9345 \]
\[ \sigma = 0.6503 \]
\[ \rho = 0.5241 \]
\[ r = 0.015 \]
\[ \beta = 0.06 \] \hspace{1cm} (15)

The long-run estimate of \( \theta \) in (15) is assumed to be the reference value for the investor. \( \theta \)-ambiguity averse investor builds the interval for \( \theta \) values \([\theta, \bar{\theta}]\) around it.

Taking expectations of the second order Taylor expansion of \( v_t = \frac{1}{y_t} \) around \( \theta \):

\[ E [v_t] \approx \frac{1}{\theta} + \frac{1}{2} \frac{\sigma^2}{\theta^2 \kappa} = \frac{1}{\theta} + \frac{\text{Var}(y_t)}{\theta^3} \] \hspace{1cm} (16)
### 5.1 Simulation: Exact Solution.

**Portfolio Choice**

#### Table 1 (with $\psi=1$)

<table>
<thead>
<tr>
<th>Expected Annual Standard Deviation of Risky Asset Return</th>
<th>19,1314%</th>
<th>20%</th>
<th>25%</th>
<th>30%</th>
</tr>
</thead>
<tbody>
<tr>
<td>implied $\Theta$</td>
<td>$\Theta = 27.935$</td>
<td>$\Theta = 25.612$</td>
<td>$\Theta = 16.604$</td>
<td>$\Theta = 11.706$</td>
</tr>
<tr>
<td>implied ambiguity level</td>
<td>0%</td>
<td>8%</td>
<td>41%</td>
<td>58%</td>
</tr>
</tbody>
</table>

#### A - Mean allocation to risky asset (%)

<table>
<thead>
<tr>
<th>R.R.A.</th>
<th>0.75</th>
<th>2.00</th>
<th>4.00</th>
<th>20.00</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>305.66</td>
<td>111.37</td>
<td>55.24</td>
<td>10.98</td>
</tr>
<tr>
<td></td>
<td>280.24</td>
<td>102.11</td>
<td>50.64</td>
<td>10.07</td>
</tr>
<tr>
<td></td>
<td>181.68</td>
<td>66.20</td>
<td>32.83</td>
<td>6.52</td>
</tr>
<tr>
<td></td>
<td>128.09</td>
<td>46.67</td>
<td>23.15</td>
<td>4.60</td>
</tr>
</tbody>
</table>

#### B - Ratio of hedging demand over myopic demand (%)

<table>
<thead>
<tr>
<th>R.R.A.</th>
<th>0.75</th>
<th>2.00</th>
<th>4.00</th>
<th>20.00</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1.19</td>
<td>-1.68</td>
<td>-2.47</td>
<td>-3.09</td>
</tr>
<tr>
<td></td>
<td>1.19</td>
<td>-1.68</td>
<td>-2.47</td>
<td>-3.09</td>
</tr>
<tr>
<td></td>
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<td>-1.68</td>
<td>-2.47</td>
<td>-3.09</td>
</tr>
<tr>
<td></td>
<td>1.19</td>
<td>-1.68</td>
<td>-2.47</td>
<td>-3.09</td>
</tr>
</tbody>
</table>

#### C - Ratio of Ambiguity demand over myopic demand (%)

<table>
<thead>
<tr>
<th>R.R.A.</th>
<th>0.75</th>
<th>2.00</th>
<th>4.00</th>
<th>20.00</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td></td>
<td>-8.41</td>
<td>-8.18</td>
<td>-8.11</td>
<td>-8.06</td>
</tr>
<tr>
<td></td>
<td>-58.79</td>
<td>-57.12</td>
<td>-56.66</td>
<td>-56.30</td>
</tr>
</tbody>
</table>
5.1 Simulation: Exact Solution.

- Comments on Table 1:
  - Portfolio choice strongly reacts to $\theta$ – ambiguity.
5.1 Simulation: Exact Solution.

- Comments on Table 1 (cont.):
  - $\theta$ - ambiguity has the same impact (direction) of risk aversion on the portfolio choice.
The solution of the ambiguity problem depends on the combination between investors risk preferences and the characterization of the investment opportunity set dynamics. In our setting, precision is always good.

- Ambiguity aversion is relevant if investor can not update continuously his portfolio. Expectation of future precision drives the risky asset demand.

- In this latter case, the risky asset demand is decomposed in three components: myopic and intertemporal hedging demand and ambiguity demand (novelty).

- It is found that ambiguity demand has a relevant empirical dimension, much higher than that of intertemporal hedging demand.

- Stochastic Precision of the risky return can be very relevant for the portfolio choice, essentially because of its ambiguity and not because of its risk.