Diversity and Arbitrage in a Regulatory Breakup Model

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1 Motivation
   - The Model
   - Diversity Implies Arbitrage in Standard Models

2 Regulatory Model
   - Regulatory Procedure
   - Examples of Regulated Market Models
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2 Regulatory Model
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Stock capitalization (total shares \times price per share) process \( X_t = (X_{1,t}, \ldots, X_{n,t}) \) is the unique strong solution to

\[
dX_{i,t} = X_{i,t} \left[ b_i(X_t) dt + \sum_{v=1}^{d} \sigma_{iv}(X_t) dW_{v,t} \right], \quad 1 \leq i \leq n.
\]

Money market account \( B \equiv 1, (r \equiv 0) \).

\( d \geq n \), and the covariance matrix \( \sigma(x)\sigma(x)' \in \mathbb{R}^{n \times n} \), is uniformly elliptic. That is, there exists \( \kappa > 0 \) such that

\[
\xi' \sigma(x)\sigma(x)'\xi \geq \kappa \|\xi\|^2, \quad \forall \xi \in \mathbb{R}^n, \forall x \in (0, \infty)^n.
\]
Diversity

- **Market weight process** $\mu$:

$$
\mu_{i,t} := \mu_i(X_t) := \frac{X_{i,t}}{\sum_{j=1}^{n} X_{j,t}}, \quad 1 \leq i \leq n.
$$

- Each $X_i$ is strictly positive, so $\mu$ lives in the simplex

$$
\Delta^n_+ := \left\{ (\pi_1, \ldots, \pi_n) \in (0, \infty)^n \mid \sum_{i} \pi_i = 1 \right\}.
$$

- Reverse order statistics notation:

$$
X(1) \geq X(2) \geq \ldots \geq X(n).
$$

**Definition**

A market is **diverse** on horizon $T$ if there exists $\delta \in (0, 1)$ such that

$$
\mu_{(1),t} < 1 - \delta, \quad \forall t : 0 \leq t \leq T.
$$
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R. Fernholz [Fer99, Fer02]: diversity and equivalent martingale measures (EMMs) are incompatible.

**Standard Model Assumptions:**

- Capitalizations are Itô processes ($\Rightarrow$ continuous paths);
- covariance is uniformly elliptic;
- continuous trading;
- no transaction costs;
- no dividends;
- number of companies is constant.

Under these assumptions diversity can be maintained only via singular repulsive down-drift of $\mu_{(1)}$ [FKK05]. Such models admit relative arbitrage with respect to the market portfolio over any horizon. These arbitrages are functionally generated from $\mu$, not requiring knowledge of $b$ or $\sigma$ to construct.
Motivating Question

Are diversity and no-arbitrage compatible if diversity is maintained by a regulator breaking up any company that becomes too large?
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Regulatory Procedure I

Confine market weights $\mu$ to $U^\mu$ by redistribution of capital via a deterministic mapping $\mathcal{R}^\mu$ upon $\mu$’s exit from $U^\mu$.

*Assumption:* Total capital is conserved.

**Definition**

A *regulation rule* $\mathcal{R}^\mu$ with respect to the open, nonempty set $U^\mu \subset \Delta^n_+$ is a Borel function

$$\mathcal{R}^\mu : \partial U^\mu \rightarrow U^\mu$$

The regulation rule is equivalently described as acting on $X$ via

$$U^x := \mu^{-1}(U^\mu) = \{x \in (0, \infty)^n \mid \mu(x) \in U^\mu\}$$

$$\mathcal{R}^x : \partial U^x \rightarrow U^x$$

$$\mathcal{R}^x(x) := \left(\sum_{i=1}^{n} x_i\right) \mathcal{R}^\mu(\mu(x))$$
$U^x$ is a conic region, i.e. $x \in U^x \Rightarrow \lambda x \in U^x$, $\forall \lambda > 0$, allowing any total market value for a given $\mu \in U^\mu$.

Regulation is first applied at the exit and stopping time

$$\alpha_1 := \inf \{ t > 0 \mid \mu_t \in \partial U^\mu \} = \inf \{ t > 0 \mid X_t \in \partial U^x \}$$

After $\alpha_1$ the capitalizations “reset” as if starting afresh from initial point $\mathcal{R}^x(X_{\alpha_1})$ until exit from $U^x$ again.

Applying this procedure inductively defines the regulated capitalization process.

To obtain a diverse regulated market, choose e.g.

$$U^\mu = \{ \pi \in \Delta^+_n \mid \pi(1) < 1 - \delta \}.$$
\( \tau_0 = 0, \quad W^1 := W, \quad X^1 = X, \quad \tau_1 := \alpha_1 := \inf \{ t > 0 \mid X^1_t \in \partial U^x \} \).

By induction define the following for \( k \geq 2 \), on \( \{ \tau_{k-1} < \infty \} \),

\[
W^k_t := W_{\tau_{k-1}+t} - W_{\tau_{k-1}}, \quad \forall t \geq 0, \quad \text{a B.M. on } \{ \tau_{k-1} < \infty \}
\]

\[
dX^k_{i,t} = X^k_{i,t} \left( b_i(X^k_t)dt + \sum_{\nu=1}^{d} \sigma_{i\nu}(X^k_t)dW^k_t \right), \quad 1 \leq i \leq n,
\]

\[
X^k_0 = \mathcal{R}^x(X^{k-1}_{\alpha_{k-1}}),
\]

\[
\alpha_k := \inf \left\{ t > 0 \mid X^k_t \in \partial U^x \right\}, \quad \tau_k := \sum_{j=1}^{k} \alpha_j.
\]

\( X^k \) is the unique strong solution to the SDE above, on \( \{ \tau_{k-1} < \infty \} \) with filtration \( \{ \mathcal{F}_{\tau_{k-1}+t} \}_{t \geq 0} \).
Regulated Capitalization Process

- There is a possibility of explosion, that is of \( \lim_{k \to \infty} \tau_k < \infty \).

\[
N_t := \sum_{k=1}^{\infty} 1_{\{t > \tau_k\}} \in \mathcal{F}_t, \quad \tau_\infty := \lim_{k \to \infty} \tau_k.
\]

**Definition**

For regulation rule \((U^\mu, \mathcal{R}^\mu)\) and initial point \(y_0 \in U^x\), the *regulated capitalization process* is defined as

\[
Y_t(\omega) := y_0 1_{\{0\}}(t) + \sum_{k=1}^{\infty} 1_{(\tau_{k-1}, \tau_k]}(\omega, t) X_{t-\tau_{k-1}}^k(\omega), \quad (\omega, t) \in [0, \tau_\infty).
\]

\[
Y_0 = y_0 = x_0 = X_0^1
\]

If \(P(\tau_\infty = \infty) = 1\), then call the triple \((y_0, U^\mu, \mathcal{R}^\mu)\) *viable*.

- The examples in this talk are viable. For the technical details, see our paper [SF10].
Split-Merge Regulation

- Split the largest company and simultaneously force the smallest two to merge.
- Let $p(i)$ return the index of the $i$th largest capitalization, e.g. $p(1) = i$, when $x_i$ is the largest of $\{x_j\}_{1}^{n}$.
- For $n \geq 3$ and any open, nonempty $U^{\mu} \subseteq \Delta_{+}^{n}$, define $\mathcal{R}^{\mu} : \partial U^{\mu} \rightarrow U^{\mu}$ via
  
  \begin{align*}
  \mu_{p(1)} & \mapsto \frac{\mu(1)}{2}, \\
  \mu_{p(n-1)} & \mapsto \frac{\mu(1)}{2}, \\
  \mu_{p(n)} & \mapsto \mu(n-1) + \mu(n), \\
  \mu_{p(i)} & \mapsto \mu_i, \quad \text{for } i \not\in \{1, n-1, n\}.
  \end{align*}

- This will be the regulatory rule used in applications, with $U^{\mu}$ to be specified later.
**Assumption:** Portfolio wealth is conserved at regulation events: $V_{\tau_k^+} = V_{\tau_k}$. Realistic for breakups and merges.

This implies that capital gains are not given by $(H \cdot Y)_t$.

Would like to represent the capital gains process as a stochastic integral.

Define a *net capitalization process* $\hat{Y}$, reflecting only the non-regulatory movements of $Y$:

$$\hat{Y}_t := Y_t - \sum_{k=1}^{N_t} \Delta Y_k, \quad \Delta Y_k := Y_{\tau_k^+} - Y_{\tau_k}.$$  

Recalling that $Y_{\tau_k^+} = \mathcal{R}^x(Y_{\tau_k}) = X_{0}^{k+1}$ on $\{\tau_k < \infty\}$, then

$$\hat{Y}_t = X_{0}^{1} + \sum_{k=1}^{N_t} (X_{\alpha_k}^{k} - X_{0}^{k}) + (X_{t - \tau_{N_t}}^{N_t+1} - X_{0}^{N_t+1}).$$
A wealth process \( V^H \) in the regulated model should be locally self-financing on \( (\tau_{k-1}, \tau_k] \), for each \( k \in \mathbb{N} \).

This combined with the assumption of wealth-conservation at \( \{\tau_k\}_{1}^{\infty} \) leads to the following definitions.

**Definition**

Admissible trading strategies are predictable processes \( H \) which are \( \hat{Y} \)-integrable, and for which there exists a constant \( K > 0 \):

\[
(H \cdot \hat{Y})_t \geq -K, \quad \text{a.s., } \forall t \geq 0.
\]

A self-financing wealth processes in the regulated model is any \( V^H \) which satisfies:

\[
V^H_t = V^H_0 + (H \cdot \hat{Y})_t \quad \forall t \geq 0.
\]
ELMMs in the Regulated Model I

FTAP

NFLVR for $\hat{Y}$ is equivalent to existence of an equivalent local martingale measure (ELMM) for $\hat{Y}$.

$\hat{Y}$ obeys the SDE

$$d\hat{Y}_{i,t} = Y_{i,t} \left( b_i(Y_t)dt + \sum_{v=1}^{d} \sigma_{i,v}(Y_t) dW_t^k \right), \quad 1 \leq i \leq n.$$ 

Since $\sigma(\cdot)$ is uniformly elliptic, there exists a market price of risk, $\theta := \sigma_t'(\sigma_t \sigma_t')^{-1} b_t$. When

$$\int_0^T |\theta(Y_t)|^2 dt < \infty, \quad \text{a.s., } \forall T > 0$$

then we may define the local martingale and supermartingale,

$$Z_t := \mathcal{E}\left( -(\theta(Y) \cdot W) \right)_t = \exp \left\{ - \left( \int_0^t \theta(Y_s) dW_s + \frac{1}{2} \int_0^t |\theta(Y_s)|^2 ds \right) \right\}$$
Theorem

If $Z$ is a martingale, then the measure $Q$ generated from $\frac{dQ}{dP} := Z_T$ is a local martingale measure for $\hat{Y}$ on $[0, T]$.

The usual tools, e.g. the Kazamaki and Novikov criteria, provide sufficient conditions for $Z$ to be a martingale.

Proposition

If $Q$ is an ELMM for $\hat{Y}$ and $\sigma$ is bounded, then $Q$ is an EMM and there is no relative arbitrage with respect to the market portfolio.

In particular, this can rule out functionally-generated relative arbitrages with respect to the market.
Standard vs Regulated: Compare/Contrast

**Standard Model**

\[ dX_t = X_t \star [b(X_t)dt + \sigma(X_t)dW_t] \]

- \( \mu_t \in \Delta_+^n \)
- \( X_t \in (0, \infty)^n \)
- \( V_t^H = V_0 + H \cdot X \)
- ELMM if \( \theta(X) \) is well-behaved
- Diversity can be maintained only through \( b \)
- Diversity and no-arbitrage not compatible

**Regulated Model**

\[ d\hat{Y}_t = Y_t \star [b(Y_t)dt + \sigma(Y_t)dW_t] \]

- \( \mu_t \in U^\mu \subseteq \Delta_+^n \)
- \( Y_t \in U^x \subseteq (0, \infty)^n \)
- \( V_t^H = V_0 + H \cdot \hat{Y} \)
- ELMM if \( \theta(Y) \) is well-behaved
- \( \sigma \) an \( b \) may both be constant and \( Y \) be diverse
- Diversity and no-arbitrage compatible
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Take a geometric Brownian motion model

\[ dX_{i,t} = X_{i,t} \left[ b_i dt + \sum_{\nu=1}^{n} \sigma_{i\nu} dW_{\nu,t} \right]. \]

Impose diversity by choosing the regulatory region

\[ U^{\mu} := \{ \pi \in \Delta^n_+ \mid \pi(1) < 1 - \delta \}. \]

Choose \( R^\mu \) to be the split-merge rule.

The resulting regulated market is viable.

\( \theta = \sigma^{-1} b \) is a constant, so \( Z \) is a martingale and NFLVR and no relative arbitrage hold.
Log-Pole Market

- A diverse market where each company behaves like a geometric Brownian motion when it is not the largest [FKK05].
- The volatility $\sigma$ is constant. The drift $b(\cdot)$ is given by

$$b_i(x) := g_i 1_{\mathcal{D}_i}(x) - \frac{c}{\delta} \frac{1_{\mathcal{D}_i}(x)}{\log((1 - \delta)/\mu_i(x))}, \quad 1 \leq i \leq n,$$

where $\{g_i\}_{1}^{n}$ are non-negative numbers, $c$ is a positive number, and when $x \in \mathcal{D}_i$, then $x_i$ is the largest of the $\{x_j\}_{1}^{n}$ with ties going to the smaller index.
- The largest company is repulsed away from the log-pole-type singularity in its drift at $1 - \delta$ in $\mu$-space.
- The market is diverse and has constant volatility, so over any horizon there are long-only relative arbitrage portfolios that are functionally generated from the market portfolio.
Blocking access to the singularity removes the arbitrage.
Choose $\delta' \in (\delta, \frac{n-1}{n+1})$ and

$$U^\mu := \{ \pi \in \Delta^n_+ \mid \pi(1) < 1 - \delta' \}.$$ 

Set $R^\mu$ as the split-merge rule. Then the regulated market is viable and $b \upharpoonright_{U^x} (\cdot)$ is bounded.

This implies that $\theta$ is bounded, the Novikov condition is satisfied, and so $Z$ is a true martingale.

The regulated market is diverse, satisfies NFLVR and no relative arbitrage.
• EMMs (with respect to $\hat{Y}$) and diversity (with respect to $Y$) are compatible in this regulatory breakup model.

• The key condition here is that $\theta \big|_{U^x} (\cdot)$ be well-behaved.

• When companies may split, diversity no longer imposes constraints on $b$.

• The assumption of constant number of companies, i.e. splits and merges occurring simultaneously, may be eliminated. No arbitrage and diversity remain compatible.

• **Future work:** Incorporate more stylized facts into equity market models. This will lead to further insights and clarifications regarding the feasibility of relative arbitrage with respect to the market portfolio.

