The Effect of Estimation in High–dimensional Portfolios

Luitgard A. M. Veraart

Joint work with Axel Gandy, Imperial College London

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Outline

1. Classical Portfolio Optimisation
2. Plug-In Strategies with Estimated Parameters
3. James-Stein-Shrinkage Applied to Strategies
4. $L_1$–Constrained Strategies - LASSO
5. Other Strategies
6. Application to Empirical Data
Optimal Portfolio Selection

- The asset prices: 1 bond $S_0(t) = e^{rt}$, $d$ risky assets

$$dS_i(t) = S_i(t)[\mu_i dt + \sum_{j=1}^{d} \sigma_{ij} dW_j(t)], \quad S_i(0) > 0, \quad i = 1, \ldots, d,$$

$r > 0$ interest rate, $\mu \in \mathbb{R}^d$ drift, $\sigma \in \mathbb{R}^{d \times d}$ volatility matrix of full rank (all constant), $W$ $d$-variate Brownian motion.

- Investor has $T > 0$ fixed time horizon, $X_0 > 0$ constant initial wealth, chooses $\pi_i(t)$ fraction of the wealth invested in the $i$th asset at time $t$, resulting in time-$t$-wealth $X_t$ with $dX_t = \sum_{i=0}^{d} \pi_i(t) X_t \frac{dS_i(t)}{S_i(t)}$.

- Investor seeks $\pi$ to maximise $V(\pi) := \mathbb{E}[\log(X_T)]$.

- Optimal solution

$$\pi^* = \Sigma^{-1}(\mu - r1),$$

where $\pi_0(t) = 1 - \sum_{i=1}^{d} \pi_i(t), \quad \pi = (\pi_1, \ldots, \pi_d)^T, \quad \Sigma = \sigma \sigma^T.$
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The Problem

- What if we need to estimate $\mu$?
- What if the number of risky assets $d \to \infty$?
Plug-in Merton Strategy with Estimated $\mu$

**General Unbiased Plug-in Estimator**

- Estimate $\mu$ by $\hat{\mu}$ and define the plug-in strategy $\hat{\pi} = \Sigma^{-1}(\hat{\mu} - r1)$.
- Assume $\hat{\pi} \sim N(\Sigma^{-1}(\mu - r1), V_0^2)$, $V_0 \in \mathbb{R}^{d \times d}$, then

\[
V(\hat{\pi}) = V(\pi^*) - \frac{T}{2} \text{trace}(\Sigma V_0^2).
\]

**Specific Plug-in Estimator**

- Observation period $[-t_{est}, 0]$ for $t_{est} > 0$.
- Set

\[
\hat{\mu}_i = \log(S_i(0)) - \log(S_i(-t_{est})) + \frac{1}{t_{est}} \sum_{j=1}^{d} \sigma_{ij}^2.
\]

- Then $\hat{\pi} \sim N(\Sigma^{-1}(\mu - r1), \Sigma^{-1}/t_{est})$.
- $V(\hat{\pi}) = V(\pi^*) - d \frac{T}{2t_{est}}$.
- There are realistic scenarios in which even $V(\hat{\pi}) \rightarrow -\infty$ as $d \rightarrow \infty$. 
James-Stein-Type Shrinkage of the Strategy

The James-Stein-Strategy

Let \( \hat{\pi} = \Sigma^{-1}(\hat{\mu} - r1), \pi^0 \in \mathbb{R}^d, a > 0 \) fixed constants. Consider

\[
\hat{\pi}^{JS,\pi^0} = \left(1 - \frac{a}{(\hat{\pi} - \pi^0)^T \Sigma (\hat{\pi} - \pi^0)}\right) (\hat{\pi} - \pi^0) + \pi^0.
\]

The Expected Utility for the JS-Strategy

Let \( \hat{\mu} \sim N(\mu, \Sigma/t_{est}), K \sim \text{Poisson}(\lambda), \lambda = (\pi^* - \pi^0)^T \Sigma (\pi^* - \pi^0)/2 \):

\[
V(\hat{\pi}^{JS,\pi^0}) = V(\hat{\pi}) + \frac{T}{2} a \left[ 2 \frac{d - 2}{t_{est}} - a \right] \mathbb{E} \left[ \frac{t_{est}}{d - 2 + 2K} \right].
\]

\( \hat{\pi}^{JS,\pi^0} \) dominates \( \hat{\pi} \) for \( 0 < a < 2(d - 2)/t_{est} \); optimal \( a = (d - 2)/t_{est} \).

Special Choices for \( \pi^0 \) and Optimal \( a \)

- \( \pi^0 = \pi^* \): \( V(\hat{\pi}^{JS,\pi^0}) = V(\pi^*) - \frac{T}{t_{est}} \).
- \( \pi^0 = \frac{\beta}{d} 1, \beta \in \mathbb{R} \): In some situations \( V(\hat{\pi}^{JS,\pi^0}) \to \infty \) as \( d \to \infty \).
**General Idea**

- Require that $\pi$ satisfies $\|\pi\|_1 = \sum_{i=1}^{d} |\pi_i| \leq c$ for a constant $c \geq 0$.
- $V(\pi) \geq \log(X_0) + rT - T\mathbb{E}\left\{ c \max_i |\mu_i - r| + \frac{c^2}{2} \max_{i,j} |\Sigma_{ij}| \right\}$.
- If $\max_i |\mu_i - r|, \max_{i,j} |\Sigma_{ij}|$ bounded, $V(\pi) \nrightarrow -\infty$ as $d \rightarrow \infty$.

**Specific Results**

For $\Sigma = \eta^2(\rho 11^T + (1 - \rho)I)$, $\eta > 0$, $0 \leq \rho \leq 1$ analytic results for

- the optimal $L_1$-constrained strategies, if $\mu$ known.
- for the $L_1$-constrained plug-in strategy as $d \rightarrow \infty$:
  - the distribution of $\#\{i : \pi_i^* \neq 0\}$, if $\rho = 0$,
  - an upper bound on $\lim_{d \rightarrow \infty} \mathbb{P}(\#\{i : \pi_i^* \neq 0\} > k)$, if $\rho > 0$. 

$L_1$-constrained Strategies - LASSO
Other Strategies and Performance for $d \to \infty$

**Other Norm Constraints**

- $L_0$-restricted strategies: no degeneration of expected utility as $d \to \infty$.

- $L_2$-restricted strategies: degeneration possible.

**Special $L_1$-Constraints**

- **1/$d$-strategy**: Strategy that invests the same amount into all stocks, i.e. $\pi_{c/d} = \frac{c}{d}1$ for some $c > 0$.

- **Equal Weighting of the most Extreme stocks (EWE)**:

$$\pi_{k_i}^{\text{EWE}} = \frac{c}{\beta d} \text{sign}(\hat{a}_{k_i}) \mathbb{I}(i \leq \beta d), \quad i = 1, \ldots, d,$$

where $\hat{a}_i = \frac{\hat{\mu}_i - r}{\Sigma_{ii}}$, $c > 0$, $\beta \in (0, 1)$ constants, $k_i$ are such that $|\hat{a}_{k_1}| > |\hat{a}_{k_2}| > \cdots > |\hat{a}_{k_d}|$. 
Example - Trading S&P500

- Stocks in S&P 500 index on 01/01/2006 having daily returns for all trading days between 2001 and 2008 (373 stocks, n=2011 trading days, daily returns).
- Specific random ordering of stocks. Allow the strategies to invest in the first \( d \) stocks of this ordering.
- \( X_0 = 1, \ r = 0.02, \) roughly \( T = 1. \)
- Use of unbiased estimators based on observed stock prices at time points 0, \( \Delta, 2\Delta, \ldots, (n - 1)\Delta: \)

\[
\hat{\mu}^{\text{data}} = \frac{1}{\Delta} \hat{\xi} + \frac{1}{2} \text{diag}(\hat{\Sigma}^{\text{data}}),
\]

\[
\hat{\Sigma}^{\text{data}}_{\mu,\nu} = \frac{1}{\Delta(n-2)} \sum_{i=0}^{n-2} \left[ R_{\mu}(i) - \hat{\xi}_\mu \right] \left[ R_{\nu}(i) - \hat{\xi}_\nu \right]
\]

for \( \mu, \nu = 1, \ldots, d \), where \( R_{\mu}(i) = \log \left( \frac{S_{\mu}((i+1)\Delta)}{S_{\mu}(i\Delta)} \right) \),

\[
\hat{\xi}_\mu = \frac{1}{n-1} \sum_{i=0}^{n-2} R_{\mu}(i).
\]
Analytic and Simulation Results

Expected utility plotted against the number $d$ of available stocks.
Out-of-Sample Performance

log($X_T$) with $T = 1$ year plotted against the number $d$ of available stocks.
Summary

Main Contributions
- Quantification of the effect of estimation in vast portfolios (unknown $\mu$ and large $d$).
- Analysis of strategies which are less affected by estimation.
- Analytic formulae for James-Stein and optimal $L_1$-constrained strategies.

Specific Conclusions
- Estimation effects must not be ignored in vast portfolios!
- Simple plug in strategies have a loss through estimation linear in $d$.
- James-Stein shrinkage performs better than simple plug in strategies.
- $L_1$-constrained strategies cannot degenerate.
- $L_1$-constrained strategies and particularly the EWE-strategy and $1/d$ strategy perform well also in out-of-sample tests.
References


Analytic Results for LASSO with $\Sigma = \eta^2 I$

Optimal $L_1$-constrained strategy with known $\mu$

Suppose $|\mu_1 - r| > |\mu_2 - r| > \ldots > |\mu_d - r|$. Then $\pi^\dagger = \frac{1}{\eta^2}(\mu - r1)$ is a solution to the $L_1$-constrained optimisation problem if $\|\pi^\dagger\|_1 \leq c$. Otherwise, the unique solution is

$$\pi^* = \frac{1}{\eta^2}(\text{sign}(\mu_1 - r)(|\mu_1 - r| - a), \ldots, \text{sign}(\mu_k - r)(|\mu_k - r| - a), 0, \ldots, 0)^T,$$

where

$$k = \min \left\{ l \in \{1, \ldots, d\} : c \leq \frac{1}{\eta^2} \sum_{i=1}^{l} (|\mu_i - r| - |\mu_{l+1} - r|) \right\},$$

$$a = \frac{1}{k} \left[ \eta^2 c - \sum_{i=1}^{k} |\mu_i - r| \right]$$

and $\mu_{d+1} = r$.

Plug-in strategy with iid normally distributed estimators $\hat{\mu}_1, \ldots, \hat{\mu}_d$

Let $c = \alpha c_d$, $c_d > 0$ norming constants from extreme value theory for folded normal distribution $FN(\mathbb{E}(\hat{\mu}_1 - r), \text{Var}(\hat{\mu}_1))$. Then

$$\#\{i : \pi_i^* \neq 0\} \xrightarrow{\mathcal{L}} K + 1 \ (d \to \infty),$$

where $K$ is a Poisson distribution with expected value $\alpha\eta^2$. 