The Bellman Equation for Power Utility Maximization with Semimartingales

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Bachelier Congress
Toronto, June 24, 2010
Basic Problem

- **Utility maximization:** given utility function $U(\cdot)$, consider

$$\max \ E \left[ \int_0^T U_t(c_t) \, dt + U_T(X_T(\pi, c)) \right]$$

over trading and consumption strategies $(\pi, c)$.

- **Aim of our study:** describe optimal trading and consumption for the (random) power utility

$$U_t(x) := D_t^{\frac{1}{p}} x^p, \quad p \in (-\infty, 0) \cup (0, 1),$$

with $D > 0$ càdlàg adapted and $E[\int_0^T D_s \, ds + D_T] < \infty$. 

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Outline

1. Problem Statement
2. Dynamic Programming
3. Bellman Equation
4. Uniqueness
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Utility Maximization Problem

- $d$ risky assets: semimartingale $R$ of stock returns, $R_0 = 0$
- spot prices $S = (\mathcal{E}(R^1), \ldots, \mathcal{E}(R^d))$
- given initial capital $x_0 > 0,$

$$u(x_0) := \sup_{(\pi, c) \in A} E \left[ \int_0^T U_t(c_t) \, dt + U_T(c_T) \right], \quad U_t(x) = D_t \frac{1}{p} x^p$$

- assume $u(x_0) < \infty$
- $\pi \in L(R)$ trading strategy, $c \geq 0$ optional consumption
- Wealth: $X_t(\pi, c) = x_0 + \int_0^t X_{s-}(\pi, c) \pi_s \, dR_s - \int_0^t c_s \, ds$
- Constraints: for each $(\omega, t),$ consider a set $0 \in \mathcal{C}_t(\omega) \subseteq \mathbb{R}^d$
- Admissibility: $(\pi, c) \in A$ if
  - $X(\pi, c) > 0, \ X_-(\pi, c) > 0$
  - $\pi_t(\omega) \in \mathcal{C}_t(\omega)$ for all $(\omega, t)$
  - $c_T = X_T(\pi, c)$. 
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Dynamic Programming

For \((\pi, c) \in \mathcal{A}\), let \(\mathcal{A}(\pi, c, t) := \{(\tilde{\pi}, \tilde{c}) \in \mathcal{A} : (\tilde{c}, \tilde{\pi}) = (c, \pi) \text{ on } [0, t]\}\).

Value process:

\[
J_t(\pi, c) := \operatorname{ess sup}_{(\tilde{\pi}, \tilde{c}) \in \mathcal{A}(\pi, c, t)} E\left[ \int_0^T U_s(\tilde{c}_s) \mu^\circ(ds) \mid \mathcal{F}_t \right]
\]

Proposition (Martingale Optimality Principle)

Let \((\pi, c) \in \mathcal{A}\) satisfy \(E[\int_0^T U_s(c_s) \mu^\circ(ds)] > -\infty\). Then

- \(J(\pi, c)\) is a supermartingale
- \(J(\pi, c)\) is a martingale if and only if \((\pi, c)\) is optimal.

→ Starting point for local description.
Dynamic Programming

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**Proposition (Opportunity Process)**

- There exists a unique càdlàg process $L$ such that for any $(\pi, c) \in A$,

\[
L_t \frac{1}{p} (X_t(\pi, c))^p = \text{ess sup}_{(\tilde{\pi}, \tilde{c}) \in A(\pi, c, t)} E \left[ \int_t^T U_s(\tilde{c}_s) \mu^o(ds) \mid F_t \right].
\]

- $L$ is special: $L = L_0 + A^L + M^L$.

**Interpretation:** $\frac{1}{p} L_t$ is the maximal amount of conditional expected utility that can be accumulated on $[t, T]$ from $1\$.

**In particular:** $L_T = pU_T(1) = D_T$. 
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Local Data I

Differential semimartingale characteristics wrt. a fixed increasing process $A$:

- characteristics $(b^R, c^R, F^R)$ of $R$ wrt. cut-off $h(x)$.
- characteristics $(b^L, c^L, F^L)$ of $L$ wrt. identity.
- $(b^{R,L}, c^{R,L}, F^{R,L})$ joint characteristics wrt. $(h(x), x')$, $(x, x') \in \mathbb{R}^d \times \mathbb{R}$.

Express consumption as fraction of wealth:

- Propensity to consume $\kappa := \frac{c}{X(\pi, c)}$.
- Wealth is a stochastic exponential: $X(\pi, \kappa) = x_0 \mathcal{E}(\pi \cdot R - \kappa \cdot t)$. 
Local Data II

Budget constraint: Based on $\mathcal{E}(Y) \geq 0 \iff \Delta Y \geq -1$:

$$X(\pi, \kappa) \geq 0 \iff \pi \in \mathcal{C}^0 := \left\{ y \in \mathbb{R}^d : F^R \big[ x \in \mathbb{R}^d : y^\top x < -1 \big] = 0 \right\},$$

Additional constraints: A set-valued process $\mathcal{C}$ in $\mathbb{R}^d$, $0 \in \mathcal{C}$,

(C1) $\mathcal{C}$ is predictable,

i.e., $\{\mathcal{C} \cap F \neq \emptyset\}$ is predictable for all $F \subseteq \mathbb{R}^d$ closed.

(C2) $\mathcal{C}$ is closed and convex.
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- **(C1)** $\mathcal{C}$ is predictable,
  
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- **(C2)** $\mathcal{C}$ is closed and convex.
Bellman Equation

**Assume:** $u(x_0) < \infty$, $\exists$ optimal strategy, constraints satisfy (C1)-(C2).

**Theorem**

- Drift rate $b^L$ satisfies $-p^{-1}b^L = \max_{k \in [0, \infty)} f(k) \frac{dt}{dA} + \max_{y \in \mathcal{C} \cap \mathcal{C}_0} g(y)$.
- Optimal propensity to consume: $\hat{\kappa} = (D/L)^{1/(1-p)}$.
- Optimal trading strategy: $\hat{\pi} \in \arg \max_{\mathcal{C} \cap \mathcal{C}_0} g$.

\[
\begin{align*}
    f(k) & := U(k) - kL, \\
    g(y) & := -L - y^\top \left( b^R + \frac{c^R L}{L} + \frac{(p-1)}{2} c^R y \right) + \int_{\mathbb{R}^d \times \mathbb{R}} x' y^\top h(x) F_{R,L}(d(x,x')) \\
    & \quad + \int_{\mathbb{R}^d \times \mathbb{R}} (L + x') \{ p^{-1} (1 + y^\top x)^p - p^{-1} - y^\top h(x) \} F_{R,L}(d(x,x')).
\end{align*}
\]
Bellman BSDE

Orthogonal decomposition of $M^L$ wrt. $R$:

\[ L = L_0 + A^L + \varphi^L \cdot R^c + W^L * (\mu^R - \nu^R) + N^L. \]

$\varphi^L \in L^2_{loc}(R^c)$, $W^L \in G_{loc}(\mu^R)$, $N^L$ local martingale such that $\langle (N^L)^c, R^c \rangle = 0$ and $M_{\mu^R}(\Delta N^L | \tilde{P}) = 0$.

Corollary

$L$ satisfies the BSDE

\[ L = L_0 - pU^* (L_) \cdot t - p \max_{C \cap C_0} g \cdot A + \varphi^L \cdot R^c + W^L * (\mu^R - \nu^R) + N^L \]

with terminal condition $L_T = D_T$, where

\[
g(y) := L_- y^\top \left( b^R + c^R \left( \frac{\varphi^L}{L_-} + \frac{(p-1)}{2} y \right) \right) + \int_{\mathbb{R}^d} (\Delta A^L + W^L(x) - \bar{W}^L) y^\top h(x) F^R(dx)
\]

\[ + \int_{\mathbb{R}^d} (L_- + \Delta A^L + W^L(x) - \bar{W}^L) \{p^{-1}(1+y^\top x)p - p^{-1} - y^\top h(x)\} F^R(dx). \]
Minimality

**Theorem (Conditions of main theorem)**

$L$ is the minimal solution of the Bellman equation.

For any special semimartingale $\ell$, $\exists!$ orthogonal decomposition

$$\ell = \ell_0 + A^\ell + \varphi^\ell \cdot R^c + W^\ell \ast (\mu^R - \nu^R) + N^\ell.$$

**Definition**

A solution of the Bellman BSDE is a càdlàg special semimartingale $\ell$,

- $\ell, \ell_\_ > 0$,
- $\exists C \cap C^0,*$-valued $\tilde{\pi} \in L(R)$ such that $g^\ell(\tilde{\pi}) = \sup_{C \cap C^0} g^\ell < \infty$,
- $\ell$ (and $\varphi^\ell, W^\ell, N^\ell, \ldots$) satisfy the BSDE.

With $\tilde{\kappa} := (D/\ell)^{1/(1-p)}$, call $(\tilde{\pi}, \tilde{\kappa})$ the strategy associated with $\ell$.

- If $R = M + \int d\langle M \rangle \lambda$ with $M$ cont. local martingale, then $\tilde{\pi}$ exists.
**Minimality**

**Theorem (Conditions of main theorem)**

L is the minimal solution of the Bellman equation.

For any special semimartingale \( \ell \), \( \exists ! \) orthogonal decomposition

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**Definition**

A solution of the Bellman BSDE is a càdlàg special semimartingale \( \ell \),

- \( \ell, \ell_- > 0 \),
- \( \exists \mathcal{G} \cap C^{0,*} \)-valued \( \tilde{\pi} \in L(R) \) such that \( g^\ell(\tilde{\pi}) = \sup_{\mathcal{G} \cap \mathcal{C}_0} g^\ell < \infty \),
- \( \ell \) (and \( \varphi^\ell, W^\ell, N^\ell, \ldots \)) satisfy the BSDE.

With \( \tilde{\kappa} := (D/\ell)^{1/(1-p)} \), call \( (\tilde{\pi}, \tilde{\kappa}) \) the strategy associated with \( \ell \).

- If \( R = M + \int d\langle M \rangle \lambda \) with \( M \) cont. local martingale, then \( \tilde{\pi} \) exists.
Theorem

Let $\ell$ be a solution of the Bellman equation with associated strategy $(\bar{\pi}, \bar{\kappa})$. Assume that $\mathcal{C}$ is convex and let

$$
\Gamma = \ell \dot{X}^p + \int \kappa_s \ell_s \dot{X}_s^p \, ds, \quad \dot{X} := X(\bar{\pi}, \bar{\kappa}).
$$

Then

- $\Gamma$ is a local martingale,
- $\Gamma$ martingale $\iff u(x_0) < \infty$ and $(\bar{\pi}, \bar{\kappa})$ is optimal and $\ell = L$. 

Contents from (N. 2009, available on ArXiv):

[a] The Opportunity Process for Optimal Consumption and Investment with Power Utility

[b] The Bellman Equation for Power Utility Maximization with Semimartingales

Selected related literature:

- Existence: Kramkov&Schachermayer (AAP99), Karatzas&Žitković (AoP03)
- Log-utility: Goll&Kallsen (AAP03), Karatzas&Kardaras (FS07), Kardaras (MF09)
- Mean-variance: Černý&Kallsen (AoP07)
- Power utility: Mania&Tevzadze (GeorgMJ03), Hu et al. (AAP05), Muhle-Karbe (Diss09)

Thanks for your attention!