Optimal Portfolio Selection under Disappointment Averse Utility

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One Solution is:
The heterogeneous preferences, especially **Gul (1991)**’s disappointment averse utility.
Utility Functions

We can define a Utility Function by assuming (1) a *certainty equivalent* and (2) a *smooth utility function*.

- **Certainty Equivalent** $\mu(F)$: the value which, if received with certainty, would be indifferent to receive random outcome with distribution $F$.

$$\text{Value}(\delta_{\mu(F)}) = \text{Value}(F)$$

where $\delta_{\mu(F)}(z) = I\{z \geq \mu(F)\}$ for $z \in \mathbb{R}$.

- **Smooth Utility Function** $u$: a real-valued function of the certainty equivalent $\mu(F) \in [0, \infty)$. 
**Expected Utility**

\( u \) is strictly increasing, strictly concave, continuously differentiable and 
\( u'(0+) = \infty, \ u'(\infty) = 0, \) and \( \mu(X) \) is defined by

\[
u(\mu(X)) = E[u(X)]
\]

for any random variable \( X \).

**Gul’s Disappointment Averse Utility**

For \( u : (0, \infty) \rightarrow [0, \infty) \), a certainty equivalent \( \mu \) associated with \( u \) is implicitly defined by

\[
u(\mu(X)) = E \left[ u(X) + (\frac{1}{A} - 1)(u(X) - u(\mu(X)))I\{X < \mu(X)\} \right]
\]

Here, \( A \in (0, 1] \) is the disappointment aversion coefficient.
Disappointment Averse Utility

Figure: Utility functions for wealth level. The solid line is for the disappointment averse utility with $A = 0.4$, $\theta = 1$ and $u(x) = \frac{1}{0.5} x^{0.5}$ and the dashed line is for the smooth utility $u(x) = \frac{1}{0.5} x^{0.5}$. 
DA Utility vs Loss-Averse(LA) Utility

Loss Averse Utility :

- *Endogenous* Certainty Equivalent(DA) vs *Exogenous* Reference Level(LA)
- Kahneman and Tversky (1979), Berkelaar *et al.* (2004), Gomes (2005)

- What is the Reference Level for LA Utility?
- Ang *et al.* (2005) : An optimal portfolio may not exist with LA utility.
- DA utility is derived from the Theoretical Axioms (Betweenness Axiom), while LA utility is not.
Why DA Utility?

- Allais Paradox of the standard expected utility model

The disappointment averse utility is consistent with Allais type behavior from replacing the independence axiom by a weaker axiom called the betweenness axiom.
Why DA Utility?

- Rabin’s Gamble Problem (Ang et al. (2005))
  Reject (-100/+110) gamble, then Reject (-1000/+any amount) gamble.

- Empirical Evidence based on market data:
History

- Ang, Bekaert and Liu (2005)
  - Find solutions of both static and dynamic portfolio choice under DA utility associated with the constant relative risk averse utility (CRRA utility).
  - Provide the explicit form of non-participation region of investors.

- Epstein and Zin (2001)
  - Consider an infinite time horizon asset pricing model with recursive framework of DA utility.
  - Using actual market data, find that DA utility provides a substantial improvement in the empirical performance of a representative agent.

- However, in my knowledge, no literature give a treatment of portfolio choice problem in the continuous time economy.
The Purpose

Optimal Portfolio Choice Problem

Provide an analytic method to solve the optimal portfolio choice problem when a disappointment averse investor want to maximize her utility on terminal wealth.

Implications

The numerical results for a special utility function, for example, the disappointment averse utility function associated with the constant relative risk averse (CRRA) utility.

- lower investment on a risky asset which partially explain the portfolio puzzle.
- the change of strategies among the time horizon.
Economy

- Complete market during finite time horizon \([0, T]\).
- Risk-free asset, \(S_0(t)\) and risky asset, \(S(t)\)

\[
\begin{align*}
  dS_0(t) &= rS_0(t)dt \\
  dS(t) &= \mu S(t)dt + \sigma S(t)dB_t
\end{align*}
\]

- \(\mu > r > 0, \sigma > 0\).
- \(\gamma = \frac{\mu - r}{\sigma}\) : the market price of risk
- \(H_t = \exp\left\{-(r + \frac{1}{2}\gamma^2)t - \gamma B_t\right\}\) : the pricing kernel process
- \(\pi_t\) : proportion of wealth invested in risky asset
- \(X_t\) : wealth process of the investor with initial wealth \(X_0 = x > 0\)

\[
dX_t = rX_tdt + (\mu - r)\pi_tX_tdt + \sigma\pi_tX_tdB_t
\]
General Problem

Problem

When an investor has the utility $u(x)$ on $\mu(X_T)$ which is the certainty equivalent of her terminal wealth $X_T$, the optimal investment problem is to find the admissible portfolio strategy $\pi_t$ such that

$$\max_{\pi_t} u(\mu(X_T))$$

subject to

$$dX_t = rX_t dt + (\mu - r)\pi_t X_t dt + \sigma \pi_t X_t dB_t, \quad X_0 = x$$

$$X(t) \geq 0, \text{ for all } t \in [0, T]$$
**Expected Utility Case**

Using the Martingale Method, we can derive the formulas for optimal wealth process and optimal portfolio process for our problem.

**Solution for Expected Utility case**

Let \( I(y) \) be the inverse of the marginal utility function \( u'(x) \) and \( \lambda \) be the unique solution of

\[
E[H_T I(\lambda H_T)] = x
\]

The optimal terminal wealth and intermediate wealth are

\[
X_T^* = I(\lambda H_T), \quad X_t^* = \frac{1}{H_t} E[H_T I(\lambda H_T) | \mathcal{F}_t]
\]

if there exists a portfolio process \( \pi^* \) satisfying that \( X_t^{X,\pi^*} = X_t^* \).

A sufficient condition to exist \( \pi^* \) is provided by Karatzas and Wang(2001).
Disappointment Averse Utility Case

$$\max_{\pi_t} u(\mu(X)) = \max_{\pi_t} E \left[ u(X) + (\frac{1}{A} - 1)(u(X) - u(\mu(X)))I\{X < \mu(X)\} \right]$$

**Step 1**

Solve the following problem given $\mu(X_T) = \theta$,

$$\max_{\pi_t} v(x, \pi; \theta) = \max_{\pi_t} E \left[ u(X_T) + (\frac{1}{A} - 1)(u(X_T) - u(\theta))I\{X_T < \theta\} \right]$$

subject to

$$dX_t = rX_t dt + (\mu - r)\pi_t X_t dt + \sigma \pi_t X_t dB_t, \quad X_0 = x$$

$$X_t \geq 0, \quad \text{for all } t \in [0, T]$$

By applying the Martingale Method, we can solve this step.
Disappointment Averse Utility Case

Step 2

Let $X^*_t(\theta)$ and $\pi^*_t(\theta)$ be the optimal wealth process and the optimal portfolio process when $\mu(X_T) = \theta$ is given. Find the fixed point $\theta$, in that the solution of the equation

$$u(\theta) = v(x, \pi^*(\theta); \theta) = E \left[ u(X^*_T(\theta)) + \left( \frac{1}{A} - 1 \right)(u(X^*_T(\theta)) - u(\theta))I\{X^*_T(\theta) < \theta} \right]$$
Example : DA Utility associated with CRRA

- The CRRA utility function is defined by for any \( x \in [0, \infty) \),

\[
u(x) := \frac{1}{\alpha} x^\alpha \quad \text{if } \alpha \leq 1 \text{ and } \alpha \neq 0
\]

where \((1 - \alpha)\) is the relative risk aversion of an investor.

- The DA utility associated CRRA utility
The certainty equivalent is defined implicitly by

\[
\frac{1}{\alpha} \mu(X)^\alpha = E \left[ \frac{1}{\alpha} X^\alpha + \left( \frac{1}{A} - 1 \right) \left( \frac{1}{\alpha} X^\alpha - \frac{1}{\alpha} \mu(X)^\alpha \right) I\{X<\mu(X)\} \right]
\]

where \( A \in (0, 1] \) is her disappointment aversion coefficient. The utility of the investor on her wealth \( X \) is \( \frac{1}{\alpha} \mu(X)^\alpha \).
Theorem 4.1 (Optimal Portfolio and Optimal Wealth)

When an investor has CRRA utility with risk aversion \((1 - \alpha)\) and her initial endowment is \(x > 0\), her optimal portfolio is

\[
\pi_t^* = \frac{\mu - r}{\sigma^2 (1 - \alpha)}, \quad \text{for } t \in [0, T]
\]

and her optimal wealth process is

\[
X_{t}^{x, \pi^*} = xH_t^{\frac{1}{\alpha - 1}} \exp \left\{ \frac{\alpha}{\alpha - 1} \left( r - \frac{\gamma^2}{2(\alpha - 1)} \right) t \right\}, \quad \text{for } t \in [0, T]
\]
Theorem 4.2 (Optimal Terminal Wealth)

Under the given value $\theta$ for the certainty equivalent of terminal wealth, the optimal wealth at terminal time $T$ corresponding to $\theta$ is

$$X_{T}^{x, \pi^{*}(\theta)} = \begin{cases} 
(\lambda^{*}(\theta)H_{T}) \frac{1}{\alpha-1} & \text{if } \lambda^{*}(\theta)H_{T} < \theta^{\alpha-1} \\
\theta & \text{if } \theta^{\alpha-1} \leq \lambda^{*}(\theta)H_{T} \leq \frac{1}{A} \theta^{\alpha-1} \\
(A\lambda^{*}(\theta)H_{T}) \frac{1}{\alpha-1} & \text{if } \frac{1}{A} \lambda^{*}(\theta)H_{T} < \theta^{\alpha-1}
\end{cases}$$
Theorem 4.2 - Continue

where \( \lambda^*(\theta) \) is the unique solution of the following equation.

\[
\begin{align*}
x &= \theta e^{-rT} \left( N\left( \frac{1}{\gamma \sqrt{T}} \ln(\lambda) + \gamma \sqrt{T} + B_1(\theta) \right) - N\left( \frac{1}{\gamma \sqrt{T}} \ln(\lambda) + \gamma \sqrt{T} + B_2(\theta) \right) \right) \\
+ &\ \lambda \frac{1}{\alpha - 1} \exp\left\{ - \frac{\alpha}{\alpha - 1} \left( r - \frac{\gamma^2}{2(\alpha - 1)} \right) T \right\} N\left( - \frac{\alpha \gamma \sqrt{T}}{\alpha - 1} - \frac{1}{\gamma \sqrt{T}} \ln(\lambda) - B_1(\theta) \right) \\
+ &\ (A\lambda) \frac{1}{\alpha - 1} \exp\left\{ - \frac{\alpha}{\alpha - 1} \left( r - \frac{\gamma^2}{2(\alpha - 1)} \right) T \right\} N\left( \frac{\alpha \gamma \sqrt{T}}{\alpha - 1} + \frac{1}{\gamma \sqrt{T}} \ln(\lambda) + B_2(\theta) \right)
\end{align*}
\]

and

\[
B_1(\theta) = - \frac{1}{\gamma \sqrt{T}} \left( (r + \frac{1}{2} \gamma^2) T + (\alpha - 1) \ln(\theta) \right), \quad B_2(\theta) = B_1(\theta) - \ln(A)
\]
Theorem 4.3 (Optimal Wealth Process)

Under same assumptions of Theorem 4.2, the investor’s optimal wealth at time $t \in [0, T)$ is

$$X_t^{x, \pi^*(\theta)} = \theta e^{-r(T-t)} \left( N\left( \frac{K_1}{\gamma \sqrt{T-t}} + \gamma \sqrt{T-t} \right) - N\left( \frac{K_2}{\gamma \sqrt{T-t}} + \gamma \sqrt{T-t} \right) \right)$$

$$+ \left( \lambda^*(\theta) H_t \right) \frac{1}{\alpha-1} e^{-\frac{\alpha}{\alpha-1} \left( r - \frac{\gamma^2}{2(\alpha-1)} \right)(T-t)} \left[ N\left( -\frac{\alpha \gamma}{\alpha-1} \sqrt{T-t} - \frac{K_1}{\gamma \sqrt{T-t}} \right) \right.$$

$$\left. + \left( A \lambda^*(\theta) H_t \right) \frac{1}{\alpha-1} e^{-\frac{\alpha}{\alpha-1} \left( r - \frac{\gamma^2}{2(\alpha-1)} \right)(T-t)} \left[ N\left( \frac{\alpha \gamma}{\alpha-1} \sqrt{T-t} + \frac{K_2}{\gamma \sqrt{T-t}} \right) \right. \right)$$

where

$$K_1 = \ln H_t + \ln \lambda^*(\theta) - (\alpha - 1) \ln \theta - \left( r + \frac{1}{2} \gamma^2 \right)(T-t), \quad K_2 = K_1 + \ln A$$
Theorem 4.3 (Optimal Portfolio Process)

Under same assumptions of Theorem 4.2, her optimal portfolio weight at time $t$ is

$$
\pi^*_t(\theta) = \frac{1}{\sigma X_t^{\pi^*_t(\theta)}} \left[ \frac{\gamma}{1 - \alpha} (\lambda^*(\theta)H_t) \right]^{\frac{1}{\alpha-1}} e^{-\frac{r}{\alpha-1} \frac{\gamma^2}{2(\alpha-1)}(T-t)}
\times \left( \frac{K_1}{\gamma \sqrt{T-t}} - \frac{\alpha \gamma}{\alpha - 1} \sqrt{T-t} \right) + A \frac{1}{\alpha-1} N\left( \frac{K_2}{\gamma \sqrt{T-t}} + \frac{\alpha \gamma}{\alpha - 1} \sqrt{T-t} \right)
- \frac{\theta}{\sqrt{2\pi(T-t)}} e^{-(r+\frac{1}{2} \gamma^2)(T-t)} \left( e^{-\frac{K_1}{2\gamma^2(T-t)}} - e^{-\frac{K_2}{2\gamma^2(T-t)}} \right)
+ \frac{(\lambda^*(\theta)H_t)}{\sqrt{2\pi(T-t)}} e^{-\frac{r}{\alpha-1} \frac{\gamma^2}{2(\alpha-1)}(T-t)} \left( e^{-\frac{\alpha}{\alpha-1} K_1 - \frac{K_1^2}{2\gamma^2(T-t)}} - A \frac{1}{\alpha-1} e^{-\frac{\alpha}{\alpha-1} K_2 - \frac{K_2^2}{2\gamma^2(T-t)}} \right)
$$
Theorem 4.4 (Level of the Certainty Equivalent)

Let $\tilde{\theta}$ be a constant defined by $\tilde{\theta} = x \exp \{ (r - \frac{\gamma^2}{2(\alpha - 1)}) T \}$. The reference rate, $\mu(X_T^{X, \pi^*})$, is a solution in $(0, \tilde{\theta})$ of the following equation.

\[
\theta^\alpha = \theta^\alpha \left( N\left( \frac{1}{\gamma \sqrt{T}} \ln(\lambda^*(\theta)) + B_1(\theta) \right) - \frac{1}{A} N\left( \frac{1}{\gamma \sqrt{T}} \ln(\lambda^*(\theta)) + B_2(\theta) \right) \right) \\
+ \left( \lambda^*(\theta) \right)^{\alpha - 1} e^{-\frac{\alpha}{\alpha - 1} (r - \frac{\gamma^2}{2(\alpha - 1)}) T} N\left( -\frac{\alpha \gamma \sqrt{T}}{\alpha - 1} - \frac{1}{\gamma \sqrt{T}} \ln(\lambda^*(\theta)) - B_1(\theta) \right) \\
+ A \frac{1}{\alpha - 1} (\lambda^*(\theta))^{\alpha - 1} e^{-\frac{\alpha}{\alpha - 1} (r - \frac{\gamma^2}{2(\alpha - 1)}) T} N\left( \frac{\alpha \gamma \sqrt{T}}{\alpha - 1} + \frac{1}{\gamma \sqrt{T}} \ln(\lambda^*(\theta)) + B_2(\theta) \right)
\]

Moreover, the optimal wealth and portfolio weight are obtained by substituting $\theta$ by the solution $\mu(X_T^{X, \pi^*})$. 
Results - Optimal Terminal Wealth

**Figure:** The optimal terminal wealth among the realized value of pricing kernel $H_T$. The solid line is for a disappointment averse investor with $A = 0.44$ and $\alpha = 0.5$ and the dashed line is for a CRRA type investor with $\alpha = 0.5$. The parameters are $r = 0.0408$, $\mu = 0.1063$, and $\sigma = 0.2193$. Investor’s initial wealth is $x = 1$ and length of life time is $T = 1$. 
Results - Optimal Intermediate Wealth

Figure: The optimal intermediate wealth among $H_t$ at time $t = 0.5$. The solid line is for a disappointment averse investor with $A = 0.44$ and $\alpha = 0.5$ and the dashed line is for a CRRA type investor with $\alpha = 0.5$. The parameters are assumed as same as in figure 2.
Results - Optimal Portfolio Amount

Figure: **The optimal portfolio amount among** $H_t$ **at time** $t = 0.5$. The solid line is for a disappointment averse investor with $A = 0.44$ and $\alpha = 0.5$ and the dashed line is for a CRRA type investor with $\alpha = 0.5$. The parameters are assumed as same as in figure 2.
Results - Optimal Portfolio Weight

Figure: The optimal portfolio weight among investor’s wealth level at time $t = 0.5$. The solid line is for a disappointment averse investor with $A = 0.44$ and $\alpha = 0.5$ and the dashed line is for a CRRA type investor with $\alpha = 0.5$. The parameters are assumed as same as in figure 2.
Results - Reference Level (Certainty Equivalent)

Figure: The reference level $\mu(X^*_T)$ among disappointment aversion coefficient $A$. We assume that all parameters except for $A$ are as same as in figure 2.
Results - Portfolio Weight among DA

Figure: The optimal initial portfolio weight among disappointment aversion coefficient $A$. Initial wealth level is $x = 1$ and all parameters except for $A$ are assumed as same as in figure 2.
Results - Portfolio Weight among Time Horizon

![Figure: The optimal initial portfolio weight among the length of life time $T$. The disappointment aversion coefficient is $A = 0.44$ and relative risk aversion is $\alpha = 0.5$. The parameters are $r = 0.0408$, $\mu = 0.1063$, and $\sigma = 0.2193$. Investor’s initial wealth is $x = 1.$]
Conclusion

- We provide an analytic method to solve the optimal investment problem for disappointment averse investors in a continuous time economy.

- The portfolio weight invested on a risk asset for a disappointment averse investor is lower than the portfolio weight for an investor who has the standard expected utility.

- The portfolio weight under the disappointment aversion model is changed among the time horizon, while it is constant under the CRRA utility model.