The Evaluation of Swing Contracts with Regime Switching

Carl Chiarella†, Les Clewlow* and Boda Kang†

†School of Finance and Economics
University of Technology, Sydney

*Lacima Group, Sydney

6th World Congress of the Bachelier Finance Society
Hilton, Toronto
June 26 2010
Plan of Talk

- Basic Swing Contracts with Make-up and Carry forward provisions
- Forward Price Curve with Regime Switching Volatility
- Setting up the Optimisation Problem
- Pentanomial Tree Approach
- Numerical Examples
- Conclusion
1 Literature Review

- Theoretical: Carmona and Touzi [2008] develop a mathematical framework for swing options viewed as nested optimal-stopping problems.


- Simulation: Ibáñez [2004] seeks to determine an approximate optimal strategy before pricing by simulation.


- Quantization: Bally et al. [2005], Bardou et al. [2007]. A quantization approach is implemented to price the Swing option without penalty.

- Pentanomial Tree: Wahab and Lee [2009]. A pentanomial tree approach is implemented to price swing options under GBM.
2 Issues Addressed in This Presentation

- Regime Switching Dynamics for the forward prices.
- Pentanomial tree approach to approximate the regime switching dynamics.
- Formulation of optimisation problem to account for make-up and carry-forward features under regime switching.
- Numerical implementations.
3 Basic Swing Contracts

- A basic swing contract is a contract for the supply of daily quantities of gas (within certain constraints) over a specified number of years at a specified set of contract prices. There is usually an annual contract quantity ($ACQ_{T_i}$).

- Each gas year there is a minimum volume of gas (Take-or-Pay or Minimum Bill) which will be charged for regardless of the actual quantity of gas taken ($MB_{T_i}$).

- Each day of the gas year there is a maximum volume of gas which can be taken. Hence each gas year there is a maximum volume of gas which can be taken ($MAX_{T_i}$).
Figure 1: The Basic Swing Contract. \( T_i \) is year \( i \), \( t_{ij} \) is \( j \)th day of year \( i \)

\[
MB_{T_i} = \alpha_i \cdot ACQ_{T_i}, \quad MAX_{T_i} \geq ACQ_{T_i} = \sum_{j=0}^{J} q_{t_{i,j}}
\]
3.1 A Basic Take-or-Pay Contract as a Strip of Call Options

- A Take-or-Pay contract can be viewed as a variable volume swap or a strip of variable volume options with constraints.

- In the absence of a Take-or-Pay constraint

  \[ \text{Minimum Bill} = 0 \]

  the optimal strategy each day is to purchase the max. allowable quantity when the market price is above the contract purchase price and nothing otherwise.

- In this case the contract has the maximum amount of flexibility and the value is equivalent to a strip of European Call options.
Figure 2: Payoff Diagram: Take-or-Pay as Strip of Call Options.
3.2 A Basic Take-or-Pay Contract as a Swap

- If
  \[ \text{Minimum Bill} = \text{Maximum Annual Quantity}, \]
  the optimal daily strategy typically is to purchase the maximum allowable quantity regardless of the market price (depends on form of penalty).

- Now the contract is equivalent to a swap and has the minimum amount of flexibility and value.
Figure 3: Payoff Diagram: Take-or-Pay as Swap
3.3 A Take-or-Pay Contract is a Combination of Call Strip and a Swap

- If

\[ 0 < \text{Minimum Bill} < \text{Maximum Annual Quantity}, \]

then the optimal strategy is to exercise like a strip of call options until the time left (to end of contract) is just sufficient to reach min. bill by taking the max. each day.

- In the constrained region there is a critical spot price (maybe less than the contract price) above which it is optimal to take the max. daily quantity, even though this results in a loss relative to the spot price case.
Figure 4: Payoff Diagram: Take-or-Pay as Combination of a Swap and Call Strip.
3.4 Swing Contracts with Make-Up and Carry Forward

- **Make-Up**
  - In years where the gas taken is less than Minimum Bill the shortfall (paid for in current year) is added to the *Make-Up Bank* ($M_{T_i}$).
  - In later years where the gas taken is greater than some reference level (typically Minimum Bill or ACQ) additional gas can be taken from the Make-Up Bank and a refund paid.

- **Carry Forward**
  - In years where the gas taken is greater than some reference level (typically ACQ) the excess gas is added to the *Carry Forward Bank* ($C_{T_i}$).
  - In later years Carry Forward Bank gas can be used to reduce the Minimum Bill for that year.
Figure 5: Carry Forward Bank

\[ Q_i = \text{quantity taken in year } i \]

\[ CB_{T_i} = \text{carry forward base in year } i \]

\[ C_{T_i} = (1 - \beta_{i-1})C_{T_{i-1}} + \max\{Q_i - CB_{T_i}, 0\} \]

[evolution of carry forward bank]
$MB_{T_i} = MB_{T_i}^{(0)} - \beta_i C_{T_i}$  \[use\ carry\-forward\ bank\ to\ reduce\ min.\ bill\]

$M_{T_i} = (1 - \gamma_{i-1}) M_{T_{i-1}} + \max(MB_{T_i} - Q_{T_i}, 0)$  \[evolution\ of\ make-up\ bank\]
4 Forward Price Curve with Regime Switching

• The stochastic or random nature of commodity prices plays a central role in the models for valuing financial contingent claims, for example, swing options on commodities and gas storage contracts.

• The observed quantity – $F(t, T)$

$$F(t, T) = \text{forward price at time } t \text{ for delivery of gas at time } T.$$ 

• Those contracts are widely traded on many exchanges with prices readily observed.

• The nearest maturity forward price is used as a proxy for the spot price.

• The longer dated contracts are used to imply the convenience yield.
Figure 7: Forward price curves of 24 different maturities from March 2008 to February 2010 with data from 29/09/2006 to 19/02/2008.
4.1 Stochastic volatility needed

- Deterministic volatility models — the volatility curve is fixed and the volatility of a specific forward price can change deterministically only with maturity.

- To properly describe the actual evolution of the volatility curve, one needs a process consisting of both deterministic and random factors.

- The drawback of diffusion models is that they cannot generate sudden and sufficiently large shifts of the volatility curve.

- Adding traditional type jump processes, for example Poisson jumps, one finds that, the frequency of the jumps is too large while the magnitude of the jumps is too small.
4.2 Regime Switching is better

- An appropriate framework for modelling the dynamics of volatilities: a class of piecewise-deterministic processes which allow volatility to follow an almost deterministic process between two random jump times.

- The simplest process in this class is the continuous-time homogeneous Markov chain with a finite number of jump times. Models with such a process approximate the actual jumps in volatility with jumps over a finite set of values.

- Hidden Markov Model (HMM) - EM Algorithm, Markov Chain Monte Carlo (MCMC) approach are able to estimate the parameters of such models.
4.3 Regime Switching Forward Price Curve

We use the following model for the forward prices in the natural gas market.

\[
\frac{dF(t, T)}{F(t, T)} = \sigma_1(t, T)dW_1(t) + \sigma_2(t, T)dW_2(t),
\]

\[
\sigma_1(t, T) = <\sigma_1, X_t> c(t) \left( e^{-<\alpha_1, X_t>(T-t)}(1 - \sigma_{l_1}) + \sigma_{l_1} \right),
\]

\[
\sigma_2(t, T) = <\sigma_2, X_t> c(t) \left( \sigma_{l_2} - e^{-<\alpha_2, X_t>(T-t)} \right),
\]

\[
c(t) = c + \sum_{j=1}^{J} (d_j(1 + \sin(f_j + 2\pi j t))).
\]
4.4 One Factor Model: An Example

- To price the gas swing contract, we consider the one factor model:

$$\frac{dF(t, T)}{F(t, T)} = <\sigma, X_t> c(t) \cdot e^{-\alpha(T-t)} dW_t,$$

where $W_t$ is a standard BM and $X_t$ is a finite state Markov Chain and $c(t) = c + \sum_{j=1}^{J} (d_j (1 + \sin(f_j + 2\pi j t)))$ captures the seasonal effect.

- Here, the spot volatility $\sigma$ will take different values depending on the state of the Markov Chain $X_t$. Consequently, the spot price will follow:

$$S(t) = F(0, t) \cdot \exp \left( \int_0^t <\sigma, X_s> c(s) \cdot e^{-\alpha(t-s)} dW_s - \frac{1}{2} \Lambda_t^2 \right),$$

where $\Lambda_t^2 = \int_0^t (<\sigma, X_s> c(s) \cdot e^{-\alpha(t-s)})^2 ds$. 
5 Pentanomial Tree Construction

- Bollen (1998) constructed a pentanomial lattice to approximate a regime switching GBM and to price both European and American options.
- Wahab and Lee (2009) extended the pentanomial lattice to a multinomial tree and studied the price of swing options under the regime switching GBM dynamics.
- To construct a discrete pentanomial lattice approximating the spot price process $S(t)$, we let $Y_t = \int_0^t < \sigma, X_s > c(s) \cdot e^{-\alpha(t-s)} dW_s$.
- We build a discrete lattice to approximate $Y_t$ first, we know that:
  $$dY_t = -\alpha Y_t dt + < \sigma, X_t > c(t) dW_t.$$
5.1 Nodes

- Assume there are only two regimes for the volatility, namely low volatility $\sigma_L$ and high volatility $\sigma_H$.

- In pentanomial tree in Figure 9, each regime is represented by a trinomial tree with one branch being shared by both regimes.

- In order to minimize the number of nodes in the tree, nodes from both regimes are merged by setting the step sizes of both regimes at a $1 : 2$ ratio.
Figure 8: The recombining of a pentanomial tree.
Figure 9: The Alternative Branching Processes for the mean reverting processes. The level where the tree switches from one branching to another depends on the attenuation parameter $\alpha$ and the time step $\Delta t$. 
The time values in the tree is $t_i = i\Delta t$, where $\Delta t$ is the time step.

The levels of $Y$ are equally spaced and have the form $Y_{i,j} = j\Delta Y$, where $\Delta Y$ is the space step.

Any node in the tree can therefore be referenced by a pair of integers $(i, j)$ that is the node at the $i$—th time step and $j$—th level.

From stability and convergence considerations, a reasonable choice for the relationship between the space step $\Delta Y$ and the time step $\Delta t$ is given by (see Wahab and Lee (2009)):

$$\Delta Y = \begin{cases} \sigma_L \sqrt{3\Delta t}, & \sigma_L \geq \sigma_H/2; \\ \frac{\sigma_H}{2} \sqrt{3\Delta t}, & \sigma_L < \sigma_H/2. \end{cases}$$
5.2 Transition probabilities

- The trinomial branching process and the associated probabilities are chosen to be consistent with the conditional drift and variance of the process.

- When the volatility is in the low regime, $\sigma = \sigma_L$, looking at the inner trinomial tree, we want to match:

\[
E[\Delta Y] = -\alpha Y_{i,j} \Delta t, \quad E[\Delta Y^2] = \sigma_L^2 \Delta t + E[\Delta Y]^2;
\]

equating the first and second moments of $\Delta Y$ in the tree we have:
\[ p_{u,i,j}^L((k + 1) - j) + p_{m,i,j}^L(k - j) + p_{d,i,j}^L((k - 1) - j) = -\alpha Y_{i,j} \Delta t / \Delta Y, \]

\[ p_{u,i,j}^L((k + 1) - j)^2 + p_{m,i,j}^L(k - j)^2 + p_{d,i,j}^L((k - 1) - j)^2 = \frac{(\sigma_L^2 \Delta t + (-\alpha Y_{i,j} \Delta t)^2)}{\Delta Y^2}, \]

together with \[ p_{u,i,j}^L + p_{m,i,j}^L + p_{d,i,j}^L = 1 \] we can obtain

\[ p_{u,i,j}^L = \frac{1}{2} \left[ \frac{\sigma_L^2 \Delta t + \alpha^2 Y_{i,j}^2 \Delta t^2}{\Delta Y^2} + (k - j)^2 - \frac{\alpha Y_{i,j} \Delta t}{\Delta Y}(1 - 2(k - j)) - (k - j) \right], \]

\[ p_{d,i,j}^L = \frac{1}{2} \left[ \frac{\sigma_L^2 \Delta t + \alpha^2 Y_{i,j}^2 \Delta t^2}{\Delta Y^2} + (k - j)^2 + \frac{\alpha Y_{i,j} \Delta t}{\Delta Y}(1 + 2(k - j)) + (k - j) \right], \]

\[ p_{m,i,j}^L = 1 - p_{u,i,j}^L - p_{d,i,j}^L. \]
• When the volatility is in high regime, \( \sigma = \sigma_H \), we will have:

\[
P^H_{u,i,j} = \frac{1}{8} \left[ \frac{\sigma_H^2 \Delta t + \alpha^2 Y_{i,j}^2 \Delta t^2}{\Delta Y^2} + (k - j)^2 - \frac{\alpha Y_{i,j} \Delta t}{\Delta Y} (2 - 2(k - j)) - 2(k - j) \right],
\]

\[
P^H_{d,i,j} = \frac{1}{8} \left[ \frac{\sigma_H^2 \Delta t + \alpha^2 Y_{i,j}^2 \Delta t^2}{\Delta Y^2} + (k - j)^2 + \frac{\alpha Y_{i,j} \Delta t}{\Delta Y} (2 + 2(k - j)) + 2(k - j) \right],
\]

\[
p^H_{m,i,j} = 1 - p^H_{u,i,j} - p^H_{d,i,j}.
\]
5.3 State prices for both regimes

- We will displace the nodes in the above simplified tree by adding the proper drifts $a_i$ which are consistent with the observed forward prices.

- For $x = L, H$ we define state prices $Q_{i,j}^x$ as the present value of a security that pay off $\$1$ if $Y = j\Delta Y$ and $X_{i\Delta t} = x$ at time $i\Delta t$ and zero otherwise.

- Hence those state prices are accumulated according to

$$Q_{0,0}^L = 1, \quad Q_{0,0}^H = 0; \text{ for lower volatility regime}$$

$$Q_{0,0}^L = 0, \quad Q_{0,0}^H = 1; \text{ for higher volatility regime}$$

$$Q_{i+1,j}^L = \sum_{j'} (Q_{i,j'}^L p_{L,L}^X + Q_{i,j'}^H p_{H,L}^X) p_{j',j}^L P(i\Delta t, (i+1)\Delta t);$$
\[ Q_{i+1,j}^H = \sum_{j'} (Q_{i,j}^L p_{L,H}^X + Q_{i,j}^H p_{H,H}^X) p_{j',j}^H P(i\Delta t, (i + 1)\Delta t); \]

- Where \( p_{x,x'}^X \) is the probabilities the Markov Chain transits from the state \( x \) to the state \( x' \) and \( p_{j',j}^L \) and \( p_{j',j}^H \) are the probabilities the spot transits from \( j' \) to \( j \) but arriving at low and high volatility regime respectively and \( P(i\Delta t, (i + 1)\Delta t) \) denotes the price at time \( i\Delta t \) of the pure discount bond maturing at time \( (i + 1)\Delta t \).

- To use the state prices to match the forward price curve we use:

\[ P(0, i\Delta t) F(0, i\Delta t) = \sum_j (Q_{i,j}^L + Q_{i,j}^H) S_{i,j}, \]

- Hence the adjustment needed to ensure the tree correctly returns the observed futures curve can be calculated.
Figure 10: Spot Price Tree which is consistent with the Seasonal Forward Curve.
6 Evaluation of Swing Contract

- Let $V_t^*(S, Q, i)$ and $q_t^*(S, Q, i)$, $t = 0, 1, \ldots, T$ be the time $t$ value and decision function of a Take-or-Pay contract when the spot price is $S$, the period-to-date consumption is $Q$ and the system is in regime $i$.


- Optimal decisions ($q_T^*(S, Q, i)$) and optimal value functions ($V_T^*(S, Q, i)$) at the maturity of the contract are as follows

$$q_T^*(S, Q, i) = \begin{cases} 1, & S > K; \\ \min(\max(MB - Q, 0), 1), & S \leq K. \end{cases}$$

$$V_T^*(S, Q, i) = (S - K)q_T^*(S, Q, i) - K \max(0, Q + q_T^*(S, Q, i) - MB).$$
• For \( t = T - 1, \cdots, 0 \), working backward in time we have:

\[
V_t^*(S, Q, i) = \max_{q \in [0,1]} \left\{ q(S - K) + e^{-rdt} \sum_{j=1}^{N} p_{ij} E^i_S [V_{t+1}(S_{t+1}, Q + q, j)] \right\};
\]

\[
q_t^*(S, Q, i) = \arg\max_q \left\{ q(S - K) + e^{-rdt} \sum_{j=1}^{N} p_{ij} E^i_S [V_{t+1}(S_{t+1}, Q + q, j)] \right\}.
\]

together with the following boundary conditions:

\[
V_t^*(S, Q_{max}, i) = 0, \quad q_t^*(S, Q_{max}, i) = 0,
\]

which means that the value function will be zero and there is no gas to use if the period to date consumption reaches the maximal quantity.
7 Numerical Examples

One year take-or-pay contract price differences when

- Volatilities: $\sigma_L = 0.5$, $\sigma_H = 1.0$;
- Mean reversion rate: $\alpha = 5$;
- Forward curve: $F(0, t) = 100$;
- Interest rate: $r = 0$;
- Contract price: $K = 100$;
- Maturity time: $T = 365$.
- Minimal Bill: $MB = 365 \times 80\% = 292$;
- Transition matrix of the hidden MC: $P = \begin{bmatrix} 0.8516 & 0.1484 \\ 0.7080 & 0.2920 \end{bmatrix}$. 
Figure 11: Part of the Pentanomial tree based on the above parameters.
Figure 12: A typical evolution of Markov Chain $X(t)$. 

Evolution of the Markov Chain $X(t)$
Figure 13: Day 0 price differences in two different regimes.
Figure 14: Day 0 decision differences in two different regimes.
Figure 15: Day 0 spot delta differences in two different regimes.
One year take-or-pay contract price differences when

- Volatilities: $\sigma_L = 0.5, \sigma_H = 1.0$;
- Mean reversion rate: $\alpha = 5$;
- Forward curve: $F(0, t) = 100$;
- Interest rate: $r = 0$;
- Contract price: $K = 100$;
- Maturity time: $T = 365$;
- Minimal Bill: $MB = 365 \times 80\% = 292$;
- Transition matrix of the hidden MC: $P = \begin{bmatrix} 0.99 & 0.01 \\ 0.01 & 0.99 \end{bmatrix}$. 
Figure 16: A typical evolution of Markov Chain $X(t)$.
Figure 17: Day 0 price differences in two different regimes.
Figure 18: Day 0 decision differences in two different regimes.
Figure 19: Day 0 spot delta differences in two different regimes.
Figure 20: Different realizations of the Markov Chains.
Figure 21: Different decisions and the spot price evolutions.
8 Conclusions

- Set up swing option contracts
- Allowed for make-up and carry-forward banks
- Regime Switching model for forward curve dynamics
- Implement the pentanomial tree approach
- Some numerical examples
- Future work
  - Hedging strategies.
References


[5] Breslin, J., Clewlow, L., Strickland, C. and van der Zee, D. Swing con-


