On Correlation and Default Clustering in Credit Markets

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Bachelier Congress, Toronto 2010
Panel A: Riskless Yield Curves

Panel B: Credit Spread Curves
Single-Name Credit Risk Pricing – What do we do?

Develop general yet tractable Markovian HJM models that

- fully incorporate information on riskless and credit spread term structures

allow different volatility structures for forward rates, that can be initialized to closely match empirical structures

credit spreads and yield curves are represented by a finite set of state variables

permit shocks to the economy to impact riskless yield curves and credit spreads
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- allow arbitrary interest rate-credit spread correlations
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Multi-Name Credit Risk Pricing – What do we do?

Extend single-name Markovian HJM models to

- multi-name infection-type models
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Using Kalman filter parameter estimates, we show the importance of

- interest rate-credit spread correlations
- default contagion
- the initial credit spread curve distribution
Let $P(t, T)$ be the price at date $t$ of a pure riskless discount bond that pays $1$ at date $T$:

$$P(t, T) = e^{-\int_t^T f(t,u)du},$$

where $f(t, u)$ represents the date-$t$ forward rate for the future time increment $[u, u + dt]$. 
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We assume

$$df(t, T) = \mu_f(t, T)dt + \sigma_f(t, T)dz_f(t) + c_f(t, T)dN_f(t),$$

given $f(0, T)$. $N_f(t)$ is independent Poisson process with intensity $\eta_f$. 

HJM Models: Riskless Dynamics
Apply Ito’s lemma for jump-diffusion processes to obtain

\[
\frac{dP(t, T)}{P(t, T)} = \left( r(t) + \frac{1}{2} \sigma_p(t, T) \sigma'_p(t, T) - \int_t^T \mu_f(t, u) du \right) dt
- \sigma_p(t, T) dz_f(t) + \left( e^{-K_p(t, T)} - 1 \right) dN_f(t),
\]

where

\[
\sigma_p(t, T) = \int_t^T \sigma_f(t, u) du,
\]

\[
K_p(t, T) = \int_t^T c_f(t, u) du.
\]
HJM Models: Risky Debt

For firm $A$ that has not defaulted prior to date $t$, we have

$$dY_A(t) = \begin{cases} 
1 & \text{with probability } \eta_A(X_t) dt \\
0 & \text{with probability } 1 - \eta_A(X_t) dt,
\end{cases}$$

where $\eta_A(X_t)$ is the default arrival intensity, and $\eta_A(X_t)$ denotes LGD.
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The date-$t$ price of a bond issued by $A$ is given by

$$\Pi_A(t, T) = V_A(t, T)1_{\tau_A > t},$$

where

$$V_A(t, T) = e^{-\int_t^T (f(t,u) + \lambda_A(t,u)) du} = P(t, T) S_A(t, T).$$

$\lambda_A(t) = \eta_A(t) \ell_A(t)$ is firm $A$’s forward credit spread, $\eta_A(t)$ is the default arrival intensity, and $\ell_A(t)$ denotes LGD.
Credit Spreads Dynamics

We assume

\[ d\lambda_A(t, T) = \mu_A(t, T) \, dt + \sigma_A(t, T) \, dz_A(t) + c_{fA}(t, T) \, dN_f(t), \quad t \leq \tau_A, \]

where

- correlation with diffusive riskless term structure:
  \[ E(dz_f(t)dz_A'(t)) = \sum_{m \times n} dt = (\rho_{ij}^A) \, dt \]

- a jump in riskless rates could transmit to shocks in the credit spreads

- \( \sigma_A(t, T) \) is predictable

- \( c_{fA}(t, T) \) is a deterministic function of time to maturity, \( T - t \)
Proposition 1: HJM Restrictions on the Drift Terms

No arbitrage implies

\[ \mu_f(t, T) = \sigma_p(t, T)\sigma'_f(t, T) - c_f(t, T)e^{-K_p(t,T)}\eta_f \]

\[ \mu_A(t, T) = \sigma_{S_A}(t, T)\sigma'_A(t, T) + \sigma_f(t, T)\Sigma^A\sigma'_{S_A}(t, T) \]

\[ + \sigma_p(t, T)\Sigma^A\sigma'_A(t, T) + g_A(t, T), \]

where

\[ g_A(t, T) = \eta_f \left( c_f(t, T)e^{-K_p(t,T)} - (c_f(t, T) + c_{fA}(t, T))e^{-(K_p(t,T)+K_{fA}(t,T))} \right). \]
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Problem with HJM models:

- In general, the dynamics are not Markovian in a small number of state variables

- To overcome this issue, we curtail the volatility structures
Markovian HJM: Volatilities and Jump-Impact Factors

- **Volatilities** are given by
  \[
  \sigma_{f_i}(t, T) = h_{f_i}(t) e^{-\kappa_{f_i}(T-t)}, \\
  \sigma_{A_j}(t, T) = h_{A_j}(t) e^{-\kappa_{A_j}(T-t)},
  \]
  where \( h_{f_i}(t) \) and \( h_{A_j}(t) \) are predictable functions.

- **Example:**
  \[
  h_{f_i}(t) = \min \left( |\tilde{h}_{f_i}(t)|, \bar{h}_{f_i} \right),
  \]
  where \( \bar{h}_{f_i} \) is a large yet finite constant, and
  \[
  \tilde{h}_{f_i}(t) = \sigma_{f_r}(t).
  \]
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  \]

- **Jump-impact factors** are of the form
  \[
  c_f(t, T) = c_f e^{-\gamma_f(T-t)}, \\
  c_{fA}(t, T) = c_{fA} e^{-\gamma_{fA}(T-t)}
  \]
Proposition 2: Markovian Models for Riskless Debt

Under volatility restrictions, we get exponential affine riskless bond prices:

\[ P(t, T) = \frac{P(0, T)}{P(0, t)} \exp \left( - \sum_{i=1}^{2} \sum_{j=1}^{m} H_{ij}(t, T) \psi_{ij}(t) - H_{3}(t, T) \psi_{3}(t) + H_{J}(t, T) \right), \]

where

\[ H_{1j}(t, T) = \frac{1}{\kappa_{f_j}^2} \left( 1 - e^{-\kappa_{f_j}(T-t)} \right), \text{ for } j = 1, \ldots, m \]
\[ H_{2j}(t, T) = -\frac{1}{(2\kappa_{f_j}^2)} \left( 1 - e^{-2\kappa_{f_j}(T-t)} \right), \text{ for } j = 1, \ldots, m \]
\[ H_{3}(t, T) = \frac{c_f}{\gamma_f} \left( 1 - e^{-\gamma_f(T-t)} \right), \]
\[ H_{J}(t, T) = \eta_{f} e^{-\frac{c_f}{\gamma_f}} \int_t^T \left( e^{\gamma_f u} e^{-\gamma_f t} - e^{\gamma_f u} \right) du. \]

The dynamics of the state variables are

\[ d\psi_{1j}(t) = (h_{f_j}^2(t) - \kappa_{f_j} \psi_{1j}(t)) dt + \kappa_{f_j} h_{f_j}(t) dz_{f_j}(t) \]
\[ d\psi_{2j}(t) = (h_{f_j}^2(t) - 2\kappa_{f_j} \psi_{2j}(t)) dt \]
\[ d\psi_{3}(t) = -\gamma_f \psi_{3}(t) dt + dN_f(t). \]
Proposition 2: Markovian Models for Risky Debt

Under volatility restrictions, and assuming RMV, the risky bond price at $t$ is $\Pi_A(t, T) = P(t, T)S_A(t, T)1_{\tau_A > t}$, where $S_A(t, T)$ is exponential affine:

$$S_A(t, T) = \frac{S_A(0, T)}{S_A(0, t)} \exp[-A_0(t, T) - \sum_{j=1}^{n} (K_{0,j}(t, T)\xi_{0,j} - K_{1,j}(t, T)\xi_{1,j})$$

$$+ \sum_{i=1}^{m} \sum_{j=1}^{n} (K_{2,ij}(t, T)\xi_{2,ij} - K_{3,ij}(t, T)\xi_{3,ij} - K_{4,ij}(t, T)\xi_{4,ij})$$

$$- K_5(t, T)\xi_5(t)].$$

The dynamics of the state variables are

$$d\xi_{0,j}(t) = (h_{A_j}^2(t) - \kappa_{A_j}\xi_{0,j}(t)) \, dt + \kappa_{A_j}h_{A_j}(t) \, dZ_{A_j}(t)$$

$$d\xi_{1,j}(t) = (h_{A_j}^2(t) - 2\kappa_{A_j}\xi_{1,j}(t)) \, dt$$

$$d\xi_{2,ij}(t) = (h_{f_i}(t)h_{A_j}(t) - (\kappa_{A_j} + \kappa_{f_i})\xi_{2,ij}(t)) \, dt$$

$$d\xi_{3,ij}(t) = (h_{f_i}(t)h_{A_j}(t) - \kappa_{f_i}\xi_{3,ij}(t)) \, dt$$

$$d\xi_{4,ij}(t) = (h_{f_i}(t)h_{A_j}(t) - \kappa_{A_j}\xi_{4,ij}(t)) \, dt$$

$$d\xi_5(t) = -\gamma f A\xi_5(t) + dN_f(t).$$
Stochastic Drivers vs State Variables

- Assume forward rates are driven by $m$ stochastic drivers, and credit spreads by $n$
  - Computational burden is limited to that of $(m + n)$-dim affine models with jumps
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- Number of state variables: \(3mn + 2(m + n + 1)\)
Stochastic Drivers vs State Variables

- Assume forward rates are driven by $m$ stochastic drivers, and credit spreads by $n$
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- Number of state variables: $3mn + 2(m + n + 1)$

- Number of state variables can sometimes be reduced:
  - $m = n = 1$: 8
  - $m = n = 1$, no jumps and constant $h(\cdot)$ functions: 2
  - $m = n = 1$, no jumps and no correlations between interest rates and credit spreads: 4
What’s the Big Deal?

Consider a HJM model with $m = n = 1$, and no jumps. Assume a 30-year time horizon.
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Standard HJM:

- Forward rates of $30 \times 12 = 360$ monthly interest rates and credit spreads need to be tracked.
- As such, the model is Markovian in 720 state variables.
- If the time partitions are refined to weeks, the number of state variables increases to $2,880$. 
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  - As such, the model is Markovian in 720 state variables.
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- Markovian HJM:
  - A maximum of 8 state variables need to be maintained, no matter what the partition.
Are the Volatility Restrictions Severe?

- Empirical evidence suggests that there could be a hump in the volatility structure of forward rates.
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- Proposition 2 can be generalized to enable humped volatility structures even when $m = n = 1$. 

\[ \sigma_f(t, T) = h_f(t) \sum_{j=1}^{k} a_j e^{-\kappa_j(T-t)}, \quad k > 1. \]
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- Proposition 2 can be generalized to enable humped volatility structures even when $m = n = 1$.

- In fact, we are able to establish arbitrary shapes:

$$\sigma_f(t, T) = h_f(t) \sum_{j=1}^{k} a_j e^{-\kappa_j (T-t)}, \quad k > 1.$$
Our models are built on different underlying stochastic processes, where the number of state variables is larger than the number of stochastic drivers.
Relationship with Duffie-Kan Affine Models

- Our models are built on different underlying stochastic processes, where the number of state variables is larger than the number of stochastic drivers.

- The drift terms of the path statistics offset spot rate volatilities in a manner that allows bond yields to be affine in the states, even though the state variables themselves do not have to be affine processes.
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As a result, the family of models we have established are very rich in structure, yet are easy to implement.
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As a result, the family of models we have established are very rich in structure, yet are easy to implement.

In that sense, our analysis complements Duffie-Kan ’96.
Empirical Evidence: Using Kalman Filter

- 1yr Treasury yield
- 5yr Treasury yield
- AMR: 1yr credit spread
- AMR: 5yr credit spread
- Lennar: 1yr credit spread
- Lennar: 5yr credit spread
Importance of Interest Rate-Credit Spread Correlations

Bond options

- K=1.05
- K=1
- K=0.95
The model already allows market-wide events to affect the riskless yield curve and credit spread curves.
Systemic Credit Shocks and Default Clustering

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- We now allow a primary firm’s default to impact the credit spread of surviving (secondary) firms.
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- We now allow a primary firm’s default to impact the credit spread of surviving (secondary) firms.

- Examples: Secondary firm
  - could carry significant debt of the primary firm,
  - may sell much of its goods to a primary firm,
  - may be in competition with the primary firm.
Credit Spread Dynamics: Secondary Firms

We assume

\[ d\lambda_B(t, T) = \mu_B(t, T)dt + \sigma_B(t, T)dz_B(t) + c_{fB}(t, T)dN_f(t) \]

\[ + \sum_{i=1}^{m_B} c_{AiB}(1 - Y_{Ai}(t))dY_{Ai}(t), \ \forall t \leq \tau_B, \]

where

- correlation with diffusive riskless term structure:
  \[ E(dz_f(t)dz'_B(t)) = \Sigma_B^{m \times n} dt = \left( \rho_{ij}^B \right) dt \]

- correlation with firm A’s diffusive term:
  \[ E(dz_A(t)dz'_B(t)) = \Sigma_{AB}^{n \times n} dt = \left( \rho_{ij}^{AB} \right) dt \]

- volatility structures are curtailed
Proposition 3: Pricing Risky Debt of Secondary Firms

The price of a risky bond issued by a secondary firm is given by
\[ \Pi_B(t, T) = V_B(t, T)1_{\tau_B > t}, \]
where \( V_B(t, T) = P(t, T)S_B(t, T) \) and
\[
S_B(t, T) = \frac{S_B(0, T)}{S_B(0, t)} e^{-B_0(t,T) - \sum_{j=1}^n (K_{0,j}(t,T)\xi_{0,j} - K_{1,j}(t,T)\xi_{1,j})}
\times e^{\sum_{i=1}^m \sum_{j=1}^n (K_{2,ij}(t,T)\xi_{2,ij} - K_{3,ij}(t,T)\xi_{3,ij} - K_{4,ij}(t,T)\xi_{4,ij})}
\times e^{-K_5(t,T)\xi_5(t)}
\times e^{\sum_{i=1}^{m_B} \left( \left( 1 - e^{-c_{A_iB}(T-t)} \right) U_{A_iB}(t) - c_{A_iB}(T-t) Y_{A_i}(t) \right)}.
\]

Here, \( B_0(t, T) = \int_t^T \int_0^t g_B(v, u) \, dv \, du \). The \( K^B \) coefficients and \( \xi^B \) state variables are defined as in Proposition 2, and
\[
U_{A_iB}(t) = \int_0^{t \land \tau_{A_i}} \eta_{A_i}(u) e^{-c_{A_iB}(t-u)} \, du, \quad \text{for } i = 1, \ldots, m_B.
\]
Importance of Default Contagion: Counterparty Risk in Insurance Contracts

\[ \rho^{A} = \rho^{B} \]

\[ \rho^{AB} \]

\[ \eta_{f} \]
Importance of Default Contagion: CDS Index Tranches

10–15

15–25

25–35
Generating Default Clustering

c_{AB} = 0

c_{AB} = 0.05

c_{AB} = 0.1
### Importance of the Initial Credit Spread Curve Distr.

<table>
<thead>
<tr>
<th>Distribution of initial credit spread curves</th>
<th>Tranche spreads</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda(0, t) = 0.05$</td>
<td>3451</td>
</tr>
<tr>
<td>$\lambda(0, 0) \sim \text{Uniform}(0.025, 0.075)$ and $\lambda(0, t) = \lambda(0, 0)$</td>
<td>3523</td>
</tr>
<tr>
<td>$\lambda(0, 0) \sim \text{Uniform}(0, 0.1)$ and $\lambda(0, t) = \lambda(0, 0)$</td>
<td>3528</td>
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<tr>
<td>$\lambda(0, t)$ incr from $\lambda(0, t) = 0.0125$ to $\lambda(0, t) = 0.0875$</td>
<td>2698</td>
</tr>
<tr>
<td>$\lambda(0, t)$ incr from $\lambda(0, t) = 0.025$ to $\lambda(0, t) = 0.075$</td>
<td>2883</td>
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<tr>
<td>$\lambda(0, t)$ incr from $\lambda(0, t) = 0.0375$ to $\lambda(0, t) = 0.0625$</td>
<td>3130</td>
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<td>3857</td>
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<tr>
<td>$\lambda(0, t)$ decr from $\lambda(0, t) = 0.075$ to $\lambda(0, t) = 0.025$</td>
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<tr>
<td>$\lambda(0, t)$ decr from $\lambda(0, t) = 0.0875$ to $\lambda(0, t) = 0.0125$</td>
<td>4947</td>
</tr>
</tbody>
</table>
Summary

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- Models have exponentially affine representations for riskless and risky bond prices.
  - Yet the variance structures need not be affine.
  - The number of state variables is decoupled from the number of stochastic drivers.
  - Allow flexible specification of correlations between interest rates and credit spreads.
  - Permit default clustering through a variety of channels (diffusive correlations, jumps, contagion effects).