CORRELATION UNDER STRESS
IN
NORMAL VARIANCE MIXTURE MODELS

Natalie Packham

Joint work with Michael Kalkbrener

6th World Congress of the Bachelier Finance Society,
Toronto, 26 June 2010
STRESS TESTS OF BANK PORTFOLIOS

• **Stress test:**
  - assessment of bank capital adequacy in a strongly adverse market environment

• Basel II; Supervisory Capital Assessment Program (SCAP; US, Q2 2009)

• Stress tests as integral part of **risk management**:
  - Basel Committee [BIS, 2009]
  - De Larosière Report [Larosière et al., 2009]
  - Geneva Report [Brunnermeier et al., 2009]
  - Turner Review, page 44, [Turner, 2009]:
    - [...] This implies that any use of VAR models needs to be buttressed by the application of stress test techniques which consider the impact of extreme movements beyond those which the model suggests are at all probable.
STRESS TESTS OF BANK PORTFOLIOS (II)

• Calculate risk measures (expected loss, value-at-risk, economic capital) and regulatory capital under adverse market conditions

• Crucial inputs of any portfolio model:
  ▶ Distribution assumption on portfolio constituents, e.g.
    ▶ normally distributed asset returns
    ▶ fat-tailed asset returns
  ▶ Dependence assumption among portfolio constituents, e.g. correlation

• Translate stress scenario into constraints on risk factors
  ▶ Here: risk factors are truncated (consistency!)

• Question: How does stress testing impact correlation?
OVERVIEW

- Normal variance mixture distribution
- Factor model of asset returns
- Stressed factor model
- Conditional asset correlation (normal, $t$)
- Asymptotic asset correlation in NVM model
- Normal variance mixture distribution
- Factor model of asset returns
- Stressed factor model
- Conditional asset correlation (normal, $t$)
- Asymptotic asset correlation in NVM model
NORMAL VARIANCE MIXTURE DISTRIBUTION

• Probability space \((\Omega, \mathcal{A}, \mathbb{P})\).

• Random vector \(X\) follows a normal variance mixture (NVM) distribution if

\[
X \sim \mu + \sqrt{WB}Z,
\]

where

\[
\begin{align*}
Z &\sim N_k(0, I_k), \\
W \geq 0 &\text{ is a random variable independent of } Z, \\
B &\in \mathbb{R}^{d \times k} \text{ and } \mu \in \mathbb{R}^d.
\end{align*}
\]

• Observe that

\[
X|(W = w) \sim N_d(\mu, w\Sigma),
\]

with \(\Sigma = BB'\).
NORMAL VARIANCE MIXTURE DISTRIBUTION (II)

Examples of NVM distributions:

<table>
<thead>
<tr>
<th>Df of $W$</th>
<th>Df of $X$</th>
</tr>
</thead>
<tbody>
<tr>
<td>constant</td>
<td>multivariate normal</td>
</tr>
<tr>
<td>inverse gamma</td>
<td>multivariate $t$</td>
</tr>
<tr>
<td>generalised inverse</td>
<td>symmetric generalised hyperbolic</td>
</tr>
<tr>
<td>Gaussian</td>
<td></td>
</tr>
</tbody>
</table>

\[\text{McNeil et al., 2005}, \quad \text{[Bingham and Kiesel, 2002]}\]
- Normal variance mixture distribution
- **Factor model of asset returns**
- Stressed factor model
- Conditional asset correlation (normal, \( t \))
- Asymptotic asset correlation in NVM model
FACTOR MODEL OF ASSET RETURNS

• Define

\[ V := \sqrt{WX} \quad A_i := \sqrt{WY_i} \]

• with **common factor** \( V \) and **asset returns** \( A_1, \ldots, A_k \),

• where
  
  ▶ \( X \sim N(0, 1) \)
  
  ▶ \( Y = N_k(0, \rho) \)
  
  ▶ \( \text{Corr}(X, Y_i) = \rho_i, \ i = 1, \ldots, k \)
  
  ▶ \( W > 0 \) with \( \mathbb{E} W < \infty \), independent of \( X, Y \).
• Normal variance mixture distribution

• Factor model of asset returns

• **Stressed factor model**

• Conditional asset correlation (normal, $t$)

• Asymptotic asset correlation in NVM model
STRESSED FACTOR MODEL

- Stressed common factor, stress level $C \leq 0$:

  $V \leq C$

- Conditional distribution:

  $P^C(B) = P(B|V \leq C), \quad B \in B(\mathbb{R})$
STRESSED FACTOR MODEL

• Stressed common factor, stress level $C \leq 0$:

\[
V \leq C
\]

• Conditional distribution:

\[
P^C(B) = P(B|V \leq C), \quad B \in \mathcal{B}(\mathbb{R})
\]

• Multivariate normal or $t$-distribution: determine

\[
\text{Corr}^C(A_i, A_j)
\]

• NVM distribution: determine

\[
\lim_{{C \to -\infty}} \text{Corr}^C(A_i, A_j)
\]
STRESSED FACTOR MODEL (II)

- \((A_1, A_2) \sim N \left[ 0, \begin{pmatrix} 0.2 & 0.12 \\ 0.12 & 0.2 \end{pmatrix} \right] \)
- \(V \leq -0.3 \) (red)
- \(\rho_1 = 0.8, \rho_2 = 0.7\)
**ASSET CORRELATIONS UNDER STRESS**

**Proposition**

Let $\mathbb{E}^C$ and $\text{Var}^C$ be the expectation and variance under $\mathbb{P}^C$, respectively. Then, in the NVM factor model,

$$\text{Corr}^C(A_i, A_j) = \frac{\rho_i \rho_j \frac{\text{Var}^C(V)}{\mathbb{E}^C(W)} + (\rho_{ij} - \rho_i \rho_j)}{\sqrt{\left(\rho_i^2 \frac{\text{Var}^C(V)}{\mathbb{E}^C(W)} + (1 - \rho_i^2)\right) \left(\rho_j^2 \frac{\text{Var}^C(V)}{\mathbb{E}^C(W)} + (1 - \rho_j^2)\right)}}.$$

**Must calculate $\text{Var}^C(V)$ and $\mathbb{E}^C(W)$ depends only on $V$ and $W$.**
ASSET CORRELATIONS UNDER STRESS

Proposition

Let $E^C$ and $\text{Var}^C$ be the expectation and variance under $P^C$, respectively. Then, in the NVM factor model,

$$
\text{Corr}^C(A_i, A_j) = \frac{\rho_i \rho_j \frac{\text{Var}^C(V)}{E^C(W)} + (\rho_{ij} - \rho_i \rho_j)}{\sqrt{\left(\rho_i^2 \frac{\text{Var}^C(V)}{E^C(W)} + (1 - \rho_i^2)\right) \left(\rho_j^2 \frac{\text{Var}^C(V)}{E^C(W)} + (1 - \rho_j^2)\right)}}.
$$

Must calculate $\frac{\text{Var}^C(V)}{E^C(W)} \sim$ depends only on $V$ and $W$. 
- Normal variance mixture distribution
- Factor model of asset returns
- Stressed factor model
- **Conditional asset correlation (normal, \( t \))**
- Asymptotic asset correlation in NVM model
ASSET CORRELATION IN A MULTIVARIATE NORMAL MODEL

Proposition

Let $V, A_1, \ldots, A_k$ be standard normally distributed (i.e., $W = 1$). Then,

$$\text{Corr}^C(A_i, A_j) = \frac{\rho_i \rho_j \text{Var}^C(V) + \rho_{ij} - \rho_i \rho_j}{\sqrt{(\rho_i^2 \text{Var}^C(V) + 1 - \rho_i^2)(\rho_j^2 \text{Var}^C(V) + 1 - \rho_j^2)}},$$

with

$$\text{Var}^C(V) = 1 - \frac{C \phi(C)}{N(C)} - \frac{(\phi(C))^2}{(N(C))^2},$$

where $\phi$ is the standard normal density and $N$ is the standard normal distribution function.
ASSET CORRELATION IN MULTIVARIATE NORMAL MODEL (II)

- 5 examples:

<table>
<thead>
<tr>
<th>Example</th>
<th>( \rho_{12} )</th>
<th>( \rho_1 )</th>
<th>( \rho_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0.6</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.8</td>
<td>0.7</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
</tr>
<tr>
<td>4</td>
<td>0.1</td>
<td>0.1</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.7</td>
<td>0.02</td>
<td></td>
</tr>
</tbody>
</table>

- Standard deviation of \( V \): 20%
ASSET CORRELATION IN MULTIVARIATE NORMAL MODEL (III)

Examples

Correlation under stress in NVM models
ASSET CORRELATION IN A $T$-DISTRIBUTED MODEL

Proposition

Let $V, A_1, \ldots, A_k$ follow a \underline{multivariate t-distribution} with parameter $\nu > 2$ denoting the degrees of freedom. Then,

$$\frac{\text{Var}^C(V)}{\mathbb{E}^C(W)} = \frac{\mathbb{E}^C(V^2) - \mathbb{E}^C(V)^2}{\mathbb{E}^C(W)} = \frac{f(\nu, C)}{g(\nu, C)},$$

with

$$f(\nu, C) := B\left(\frac{\nu}{C^2 + \nu}; \frac{\nu - 2}{2}, \frac{3}{2}\right) - \frac{4 \left(\frac{\nu}{C^2 + \nu}\right)^{\nu - 1}}{(\nu - 1)^2 B\left(\frac{\nu}{C^2 + \nu}; \frac{\nu}{2}, \frac{1}{2}\right)}$$

and

$$g(\nu, C) := \frac{B\left(\frac{1}{2}, \frac{\nu}{2}\right)}{\nu - 2} - \frac{\left(\mathbb{B}\left(\frac{\nu - 2}{2}, \frac{1}{2}\right) - \mathbb{B}\left(\frac{\nu}{\nu + C^2}; \frac{\nu - 2}{2}, \frac{1}{2}\right)\right)}{\nu - 1},$$

where $\mathbb{B}(y; a, b) := \int_0^y t^{a-1} (1 - t)^{b-1} \, dt$ is the incomplete beta function and $\mathbb{B}(a, b) := \mathbb{B}(1; a, b)$ is the beta function.
Left: $\nu = 4$; right: $\nu = 10$. 

Examples: select among $1$, $2$, $3$, $4$, $5$. 

Correlation under stress in NVM models
● Normal variance mixture distribution

● Factor model of asset returns

● Stressed factor model

● Conditional asset correlation (normal, $t$)

● **Asymptotic asset correlation in NVM model**
ASYMPTOTIC ASSET CORRELATION IN NVM MODEL

- Asymptotic asset correlation depends on distribution tail
  ➞ Extreme Value Theory
EXTREME VALUE THEORY

**Theorem (Fisher-Tippett Theorem)**

Let \((X_n)\) be a sequence of iid random variables, and let 
\[ M_n = \max(X_1, \ldots, X_n). \]
If there exist norming constants \(c_n > 0, d_n \in \mathbb{R}\) and some non-degenerate distribution function \(H\) such that

\[
\frac{M_n - d_n}{c_n} \xrightarrow{\mathcal{L}} H, \quad \text{as } n \to \infty, \tag{1}
\]

then \(H\) belongs to the type of one of the following three distribution functions:

**Fréchet:**

\[
\phi_{\alpha}(x) = \begin{cases} 
0, & x \leq 0 \\
\exp\{-x^{-\alpha}\}, & x > 0 
\end{cases}, \quad \alpha > 0.
\]

**Weibull:**

\[
\psi_{\alpha}(x) = \begin{cases} 
\exp\{-(x)^{\alpha}\}, & x \leq 0 \\
1, & x > 0 
\end{cases}, \quad \alpha > 0.
\]

**Gumbel:**

\[
\Lambda(x) = \exp\{-\exp^{-x}\}, \quad x \in \mathbb{R}.
\]
EXTREME VALUE THEORY (II)

EV densities ($\alpha = 1$)

Fréchet
Weibull
Gumbel

Graph showing the EV densities for $\alpha = 1$, with Fréchet, Weibull, and Gumbel distributions plotted on the graph. The x-axis is labeled as $x$.
EXTREME VALUE THEORY (III)

**Definition (Maximum domain of attraction)**

A random variable $X$ with distribution function $F$ belongs to the **maximum domain of attraction (MDA) of $H$** if there exist constants $c_n > 0$, $d_n \in \mathbb{R}$ such that Equation (1) holds, written $X \in \text{MDA}(H)$ and $F \in \text{MDA}(H)$. 

24
Lemma

*If* $W$ *is in* $MDA(\Phi_{\alpha/2})$, *then* $V$ *is in* $MDA(\Phi_{\alpha})$.

Proposition

*Let* $W \in MDA(\Phi_{\alpha/2})$, $\alpha > 2$. *Then*

$$\lim_{C \to -\infty} \frac{\text{Var}^C(V)}{\mathbb{E}^C(W)} = \frac{1}{\alpha - 1}.$$
V IN THE FRÉCHET MDA (II)

Sketch of proof:

(i) (short)
\[ \lim_{C \to -\infty} \frac{\text{Var}^C(V)}{C^2} = \frac{\alpha}{(\alpha - 2)(\alpha - 1)^2}. \]

(ii) (lengthy)
\[ \lim_{C \to -\infty} \frac{\mathbb{E}^C(W)}{C^2} = \frac{\alpha}{(\alpha - 2)(\alpha - 1)}. \]

(Regular variation, Karamata Theorem)
V IN THE FRÉCHET MDA (III)

\[ \lim_{C \to -\infty} \text{Corr}_C(A_1, A_2) \]

![Graph showing correlation under stress in NVM models](image)

Correlation under stress in NVM models
V IN THE GUMBEL MDA

**Proposition**

Let \( V \in MDA(\Lambda) \). Then

\[
\lim_{C \to -\infty} \frac{\text{Var}^C(V)}{\mathbb{E}^C(W)} = 0.
\]

(Representation Theorem, mean excess function)

Example Correlation under stress in NVM models
**V IN THE GUMBEL MDA**

**Proposition**

Let $V \in MDA(\Lambda)$. Then

$$\lim_{C \to -\infty} \frac{\text{Var}^C(V)}{E^C(W)} = 0.$$ 

**Sketch of proof:** Show that

$$\lim_{C \to -\infty} \frac{\text{Var}(V \mid V \leq C)}{E(V^2 - C^2 \mid V \leq C)} = 0.$$ 

and that

$$\lim_{C \to -\infty} \frac{E(V^2 - C^2 \mid V \leq C)}{2E(W \mid V \leq C)} = 1.$$ 

(Representation Theorem, mean excess function)
REFERENCES

Semi-parametric modelling in finance: theoretical foundations.

BIS (2009).
Principles for sound stress testing practices and supervision.

The fundamental principles of financial regulation.
*Geneva Reports on the World Economy, 11.*

Larosière, J. et al. (2009).
Report.
*The High-Level Group on Financial Supervision in the EU.*

*Quantitative Risk Management.*
Princeton University Press, Princeton, NJ.

The Turner Review — A Regulatory Response to the Global Banking Crisis.
*FSA, March.*
Thank you for your attention!

Jun.Prof. Dr. Natalie Packham
Centre for Practical Quantitative Finance
Frankfurt School of Finance & Management
Sonnemannstr. 9-11
60314 Frankfurt am Main
n.packham@frankfurt-school.de
**EMPIRICAL EXAMPLE**

- **DAX data**, daily log returns, 5.4.1991-6.11.2009
  (Source: Reuters)

- **Empirical correlations** of asset pairs:

<table>
<thead>
<tr>
<th>Assets</th>
<th>$\rho_i$</th>
<th>$\rho_j$</th>
<th>$\rho_{ij}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>BASF, Merck</td>
<td>0.73</td>
<td>0.36</td>
<td>0.30</td>
</tr>
<tr>
<td>Daimler, Telekom</td>
<td>0.76</td>
<td>0.68</td>
<td>0.43</td>
</tr>
<tr>
<td>Daimler, Kali &amp; Salz</td>
<td>0.76</td>
<td>0.41</td>
<td>0.37</td>
</tr>
</tbody>
</table>

- **Sample sizes** of truncated DAX returns:

<table>
<thead>
<tr>
<th>Truncation level $C$</th>
<th>0%</th>
<th>−0.5%</th>
<th>−1%</th>
<th>−1.5%</th>
<th>−2%</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. of samples</td>
<td>1258</td>
<td>526</td>
<td>183</td>
<td>74</td>
<td>34</td>
</tr>
</tbody>
</table>
EMPIRICAL EXAMPLE (II)

- DAX data (Source: Reuters), daily log returns, 5.4.1991-6.11.2009
- 95% confidence intervals computed using Fisher z-transform
EMPIRICAL EXAMPLE (III)

- DAX data (Source: Reuters), daily log returns, 5.4.1991-6.11.2009
- 95% confidence intervals computed using Fisher z-transform
**EMPIRICAL EXAMPLE (IV)**

- **DAX data (Source: Reuters), daily log returns, 5.4.1991-6.11.2009**
- **95% confidence intervals computed using Fisher z-transform**