CIID default models and implied copulas

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Matthias Scherer
HVB-Institute for Mathematical Finance
Technische Universität München

Joint work with Jan-Frederik Mai and Rudi Zagst.
Aims and Agenda

1.) Present a unified framework for CIID models
2.) Axiomatically define desirable statistical properties
3.) Review classical models
4.) Present new models
Motivation: CDO pricing

- **Situation:**
  > Portfolio of $d$ credit-risky assets, $(\tau_1, \ldots, \tau_d)'$ vector of random default times.
  > $L_t := \frac{1}{d} \sum_{k=1}^{d} 1_{\{\tau_k < t\}}$, i.e. percentage of defaults up to time $t \geq 0$.

- **Two major problems:**
  1. Typically $d = 125$, i.e. large.
  2. Default times are dependent.

- **Pricing of CDOs without simulation:**
  > Assumptions: Constant and identical recovery rates, equal portfolio weights.
  > Pricing of CDOs requires:
    $$\mathbb{E}[f(L_t)] = \int_{[0,1]} f(x) \mathbb{P}(L_t \in dx), \quad f \text{ complicated (collar-type).}$$
Modeling \((\tau_1, \ldots, \tau_d)\)′: Model philosophies

- **Structural models:**
  i.) Model correlated asset processes, ii.) determine \((\tau_1, \ldots, \tau_d)\)′, and iii.) \(\{L_t\}_{t \geq 0}\).
  - 😊: Economic interpretation of (correlated) defaults.
  - 😞: Distribution \(\mathbb{P}(L_t \in dt)\) very difficult to obtain.

- **Bottom-Up models:** i.) Model \((\tau_1, \ldots, \tau_d)\)′, ii.) compute \(\{L_t\}_{t \geq 0}\).
  - 😊: (Intuitive) model for dependence between firms.
  - 😞: Distribution \(\mathbb{P}(L_t \in dt)\) difficult to obtain.

- **Top-Down models:** i.) Model \(\{L_t\}_{t \geq 0}\) directly.
  - 😊: Distribution \(\mathbb{P}(L_t \in dt)\) tractable.
  - 😞: Dependence structure between firms?
CIID default models

- **Definition (CIID model):**
  \[ \tau_k := \text{function}(M, \epsilon_k), \]
  > \( M \) is a random object (market risk factor),
  > \( \epsilon_1, \ldots, \epsilon_d \) are i.i.d. and independent of \( M \) (idiosyncratic risk factors).

- **Consequences (simple Bottom-Up model, dependence via \( M \)):**
  - **ございます**: Restrictive assumptions, e.g.
    > All default times have the same distribution \( \mathbb{P}(\tau_k \leq t) =: F(t) \).
    > The dependence structure is „very special“: conditionally i.i.d. (CIID).
  - **ございません**: Large portfolio assumption:
    > The model approximates a related Top-Down model.
    > Closed-form approximation of portfolio-loss distribution / CDO prices.
CIID models: general framework

Lemma (Unified framework):

- All CIID models can be constructed as follows:
  
  (1) Let \( \{ F_t \}_{t \geq 0} \) be càdlàg, ↗, with \( F_0 = 0 \), and \( \lim_{t \to \infty} F_t = 1 \) (\( \forall \omega \in \Omega \)).

  (2) Given \( \sigma(F_t : t \geq 0) \), let \( \tau_1, \ldots, \tau_d \) be i.i.d. with cdf \( t \mapsto F_t \).

Lemma (Canonical construction):

- Define \( (\tau_1, \ldots, \tau_d)' \) via:
  \[
  \tau_k := \inf \{ t > 0 : F_t \geq U_k \},
  \]
  where \( U_1, \ldots, U_d \sim \text{Uni}(0, 1) \) are i.i.d. and independent of \( \{F_t\}_{t \geq 0} \).

Consequence:

- A CIID model is basically a model for the market frailty \( \{F_t\}_{t \geq 0} \).
Large homogeneous portfolio approximation

Lemma (Portfolio-loss distribution):

- The distribution of the portfolio loss is available but numerically critical:
  \[ P\left(L_t = \frac{k}{d}\right) = \binom{d}{k} \mathbb{E}\left[F_t^k \left(1 - F_t\right)^{d-k}\right], \quad k = 0, 1, \ldots, d. \]

- Gliwenko-Cantelli:
  \[ P\left(\lim_{d \to \infty} \sup_{t \geq 0} |F_t - L_t| = 0\right) = 1. \]

- For \( d \gg 2 \), this justifies:
  \[ \mathbb{E}[f(L_t)] = \int_{[0,1]} f(x) \mathbb{P}(L_t \in dx) \approx \int_{[0,1]} f(x) \mathbb{P}(F_t \in dx). \]

- Equivalent to using a Top-Down model with \( L_t := F_t \).
Examples

- Model input: Term structure of default probabilities \( t \mapsto F(t) := \mathbb{P}(\tau_1 \leq t) \).

1.) **Gaussian copula model:** [Vasicek 1987, Li 2000]

\[
\tau_k := F^{-1}\left(\Phi\left(\sqrt{\rho}M + \sqrt{1-\rho}\epsilon_k\right)\right)
\]

\[
F_t := \Phi\left(\frac{\Phi^{-1}(F(t)) - \sqrt{\rho}M}{\sqrt{1-\rho}}\right), \quad t \geq 0,
\]

where \( M, \epsilon_1, \ldots, \epsilon_d \) are i.i.d. standard normal.

2.) Generalization to **infinitely divisible distributions:** [Albrecher et al. 2007]

\[
F_t := H_{[1-\rho]}^{-1}\left(H_{[1]}^{-1}(F(t)) - M\right), \quad t \geq 0,
\]

where \( H_{[t]} = \text{cdf of } X_t \) for some suitable Lévy process \( \{X_t\}_{t \in [0,1]} \) and \( M := X_\rho \).

> Corresponds to replacing normal distribution by ID distribution.

> Special cases: [Guégan, Houdain 2005], [Kalemanova et al. 2007], ...
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Desirable properties (Sep) and (Cop)

Definition (Sep), (Cop):

- (Sep) :⇔ $\mathbb{E}[F_t] = F(t) = \mathbb{P}(\tau_k \leq t)$ for all $t > 0$.

  **Advantage**: The marginal distribution $t \mapsto F(t)$ is model input.

- (Cop) :⇔ $\exists$ explicit expression for:
  $$\mathbb{P}(\tau_1 \leq t_1, \ldots, \tau_d \leq t_d) = \mathbb{E}[F_{t_1} \cdots F_{t_d}], \quad t_1, \ldots, t_d \geq 0.$$

- If both (Sep) and (Cop) hold, then one finds the (survival) copula:
  $$C(u_1, \ldots, u_d) = \mathbb{E}[F_{F^{-1}(u_1)} \cdots F_{F^{-1}(u_d)}], \quad u_1, \ldots, u_d \in [0, 1],$$
  $$\hat{C}(u_1, \ldots, u_d) = \mathbb{E}[(1 - F_{F^{-1}(1-u_1)}) \cdots (1 - F_{F^{-1}(1-u_d)}), \quad u_1, \ldots, u_d \in [0, 1].$$

- **Copula models**: dependence structure $\oplus$ marginal default probabilities.
Desirable properties
(Exc)

- **Stylized facts:** Excess clustering, multiple defaults.
  
  > Excess clustering ↔ „fast growth“ of \( \{F_t\}_{t \geq 0} \).
  
  > Multiple defaults ↔ jumps of \( \{F_t\}_{t \geq 0} \) ↔ singular component of \( C, \hat{C} \).

- **Definition (Exc):**
  
  \( (Exc) \Leftrightarrow \{F_t\}_{t \geq 0} \) exhibits jumps \( \Leftrightarrow \mathbb{P}(\tau_1 = \ldots = \tau_k) > 0 \).
Desirable properties
(Fs)

**Definition (Fs):**

- **(Fs⊖):** Static source of frailty: $F_t$ is $\cap_{u>0} \sigma(F_s : 0 \leq s \leq u)$-measurable.
  
  > E.g. $F_t := \text{function}(M, t)$, where $M$ is a random parameter.

  > Unintuitive model, since no time-evolution.

  > Typically $t \mapsto F_t$ smooth function (no jumps, no time-varying slopes).

- **(Fs⊙):** Dynamic frailty with **time-homogeneous innovations**.
  
  > E.g. $\{F_t\}_{t\geq0}$ driven by Lévy process (no stoch. vol.).

- **(Fs⊕):** Dynamic frailty with **time-inhomogeneous innovations**.
Desirable properties (Tdc)

- **Coefficient of lower-tail dependence:**
  \[
  \lambda_l := \lim_{t \downarrow 0} \mathbb{P}(\tau_i \leq t \mid \tau_j \leq t) = \lim_{t \downarrow 0} \frac{\mathbb{E}[F_t^2]}{\mathbb{E}[F_t]}.
  \]

  > Measures the likelihood of joint early defaults.

  > Empirical studies suggest that (Tdc)-supporting models are more successful in explaining CDO quotes.

  > **Attention:** Only bivariate margins considers, higher-order effects neglected!

- **Definition (Tdc):**
  \[(Tdc) :\iff \lambda_l > 0.\]
Desirable properties

(\text{Den})

\begin{itemize}
  \item \textbf{Density of the portfolio-loss approximation:}
    \begin{itemize}
      \item Implementation requires viable distribution of $F_t$ (for all $t > 0$).
      \item Sometimes, the density is only available through Laplace-inversion techniques.
    \end{itemize}
  \item \textbf{Definition (Den)}:
    
    (\text{Den}) \iff the density of $F_t$ is known explicitly for all $t > 0$.
\end{itemize}
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Examples - cont.

- Model input: Term structure of default probabilities $t \mapsto F(t) := \mathbb{P}(\tau_1 \leq t)$.

3.) **Archimedean copula model:** [Schönbucher 2002]

$$\tau_k := \inf \{ t \geq 0 : U_k \leq 1 - \exp\left(-M \varphi^{-1}(1 - F(t))\right) \},$$
$$F_t := 1 - \exp\left(-M \varphi^{-1}(1 - F(t))\right), \quad t \geq 0,$$

$M > 0$ a positive random variable with Laplace transform $\varphi$, $U_k \sim \text{Uni}(0, 1)$.

4.) **Lévy-frailty model:** [Mai, Scherer 2009]

$$\tau_k := \inf \{ t \geq 0 : M_t \geq E_k \}, \quad E_k \sim \text{Exp}(1),$$
$$F_t := 1 - e^{-M_t}, \quad M_t := \Lambda_{-\log(1-F(t))/\Psi(1)}, \quad t \geq 0,$$

with $\Lambda = \{\Lambda_t\}_{t \geq 0}$ a Lévy subordinator with $\text{LE } \Psi(x) = -\log \mathbb{E}[\exp(-x\Lambda_1)]$. 

Mai, Scherer, Zagst: CIID default models and implied copulas
5.) **Intensity based model:** [Duffie, Gârleanu 2001]

> In their general form not CIID models, but in the following special case:

\[
\tau_k := \inf \{ t \geq 0 : M_t \geq E_k \}, \quad E_k \sim \text{Exp}(1),
\]

\[
F_t := 1 - e^{-M_t}, \quad M_t := \int_0^t \lambda_s \, ds, \quad t \geq 0,
\]

where \( \{\lambda_t\}_{t \geq 0} \) is a basic affine process, i.e.

\[
d\lambda_t = \kappa (\theta - \lambda_t) \, dt + \sigma \sqrt{\lambda_t} \, dB_t + dZ_t, \quad \lambda_0 > 0.
\]

> Default probabilities are not model input. However,

\[
F(t) = 1 - \mathbb{E}[e^{-M_t}] = 1 - \exp(\alpha(1, t) + \beta(1, t) \lambda_0),
\]

for explicit functions \( t \mapsto \alpha(1, t), \beta(1, t) \), see [Duffie, Kan 1996].
Properties of the models

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<thead>
<tr>
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A new model based on Archimax copulas

Combining [Schönbucher 2002] and [Mai, Scherer 2009]:

- Define
  \[ F_t := 1 - e^{-M_t}, \quad M_t := \Lambda \bar{M} \varphi^{-1}(1-F(t))/\Psi(1), \quad t \geq 0, \]
  where \( \Lambda \) as before and \( \bar{M} > 0 \) with Laplace trafo \( \varphi \), independent of \( \Lambda \).

> Combination of Archimatedean and Marshall-Olkin copulas (\( \subset \) Archimax copulas).

- Inherits benefits from / extends earlier approaches.

- (Den) is lost (only known for special cases, Laplace-inversion required).
A new model based on Archimax copulas

Properties:

• It can be shown that \((\tau_1, \ldots, \tau_d)' \sim \hat{C}(F, \ldots, F)\) with

\[
\hat{C}(u_1, \ldots, u_d) = \varphi\left(\frac{1}{\Psi(1)} \sum_{i=1}^{d} \varphi^{-1}(u(i)) \left(\Psi(i) - \Psi(i - 1)\right)\right),
\]

where \(u(1) \leq \ldots \leq u(d)\) denotes the ordered list of \(u_1, \ldots, u_d \in [0, 1]\).

Special case (Clayton mixed with Cuadras-Augé):

> \(\Lambda\) Poisson process with intensity \(\beta > 0\).

> \(\bar{M} \sim \Gamma(1, 1/\theta)\).

> Then, it can be shown that

\[
\Pr(\Lambda_{\bar{M}t} = k) = \frac{(t \beta)^k}{\Gamma(1/\theta) k!} \left(\frac{1}{1 + \beta t}\right)^{k + \frac{1}{\theta}} \Gamma\left(k + \frac{1}{\theta}\right), \quad k \in \mathbb{N}_0.
\]
A new model based on a CGMY-type frailty

Combining [Duffie, Gârleanu 2001] and [Mai, Scherer 2009]:

- Define

\[ F_t := 1 - e^{-M_t}, \quad M_t := \Lambda \int_0^t \lambda_s ds / \Psi(1), \quad t \geq 0, \]

where \( \Lambda \) and \( \lambda \) are given as before.

> Time-varying intensity + jumps (see stoch.vol. + jumps in asset models).

😊 Incorporates most stylized facts, intuitive and flexible model.

😢 (Den) is lost (Laplace-inversion required), (Cop) is lost, (Sep) „partially“ lost.
A new model based on a CGMY-type frailty
A new model based on a CGMY-type frailty

Properties:

• Lévy subordinator $\Lambda$ accounts for clustering, e.g.

$$\mathbb{P}(\tau_1 = \ldots = \tau_k) = \frac{\sum_{i=0}^{k} \binom{k}{i} (-1)^{i+1} \Psi(i)}{\Psi(k)}, \quad k = 2, \ldots, d.$$  

• Lévy subordinator $\Lambda$ also accounts for tail-dependence:

$$\lambda_l = 2 - \frac{\Psi(2)}{\Psi(1)}.$$  

• Intensity $\lambda$ accounts for time-inhomogeneous distribution of the clusters.

• Required density must be obtained from the following Laplace trafo:

$$\mathbb{E} \left[ \exp \left( - q \int_0^t \lambda_s \, ds / \Psi(1) \right) \right] = e^{\alpha \left( \Psi(q)/\Psi(1), t \right) + \beta \left( \Psi(q)/\Psi(1), t \right) \lambda_0}, \quad q \geq 0,$$

where $\alpha(z, t), \beta(z, t)$ are known in closed form from [Duffie, Kan (96)].
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Conclusion

A unified framework and a canonical construction for CIID models is given.

Desirable statistical properties are identified and axiomatically defined.

Existing models are analyzed in this regard.

Two new models are presented.


Example: Gaussian copula model [Li 2000]

\[ F_t = \Phi \left( \Phi^{-1}(F(t)) - \sqrt{\rho} M \right) \frac{1}{\sqrt{1 - \rho}} , \quad F(t) = \mathbb{E}[F_t] = 1 - \exp(-0.03t) \]
A Lévy-based model [Mai, Scherer 2009]

\[ F_t = 1 - (1 - \rho)^{N_{0.03}t}, \quad N_1 \sim \text{Poi}(1/\rho), \quad F(t) = \mathbb{E}[F_t] = 1 - \exp(-0.03 t) \]