GARCH Intensity Models for Asset Price and Their Application to Option Valuation

Geon Ho Choe  Kyungsub Lee

Department of Mathematical Sciences
KAIST

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Outline

- Empirical studies
- Introduction to intensity model
- GARCH for intensity
- Estimation result
- Conclusion
Absence of serial correlation (left).

\[ \text{Corr}(X_t, X_{t-\ell}) \]

\[ \text{Corr}(|X_t|, |X_{t-\ell}|) \]

\( X_t \): the log-return at \( t \).
S&P 500 return series (1990 - 2009)

- Absence of serial correlation (left).
- Volatility clustering (right).

$X_t$: the log-return at $t$. 

$$\text{Corr}(X_t, X_{t-\ell})$$

$$\text{Corr}(|X_t|, |X_{t-\ell}|)$$
Leverage effect

- Negative response of $|X_t|$ to $X_{t-\ell}$, $\ell > 0$ (left).

Corr($|X_t|$, $X_{t-\ell}$)

Corr($X_t$, $|X_{t-\ell}|$)
Leverage effect

- Negative response of $|X_t|$ to $X_{t-\ell}$, $\ell > 0$ (left).
- On the other hand, $\text{Corr}(X_t, |X_{t-\ell}|)$ is negligible (right).
Even though $\text{Corr}(X_t, X_{t-\ell})$ and $\text{Corr}(X_t, |X_{t-\ell}|)$, for $\ell > 0$ are insignificant, we have non-negligible correlation between current return and past (magnitude of) return depending on the condition of $\text{sign}(X_t)$. 
Conditional correlation

Leverage effect captured by correlation on the condition of current return’s sign:

\[ \text{Corr}(X_t, X_{t-\ell} | X_t > 0) \]

\[ \text{Corr}(X_t, X_{t-\ell} | X_t < 0) \]
Conditional correlation (2)

On the condition of current return’s sign:

\[
\text{Corr}(X_t, |X_{t-\ell}| \mid X_t > 0)
\]

\[
\text{Corr}(X_t, |X_{t-\ell}| \mid X_t < 0)
\]
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\[ \text{Corr}(X_t, |X_{t-\ell}| \mid X_t < 0) \]

The price is less affected by the previous information \(|X_{t-\ell}|\) when the price decreases than the case when the price increases.
The asset price $S(t)$ satisfies

$$S(t) = S(0) \exp \left\{ \delta (N_+(t) - N_-(t)) \right\}$$

for some constant $\delta > 0$. 

Kyungsub Lee

GARCH intensity 8/23
Intensity model

The asset price $S(t)$ satisfies

$$S(t) = S(0) \exp \{ \delta (N_+(t) - N_-(t)) \}$$

for some constant $\delta > 0$.

- $(N_\pm(t) - N_\pm(t_{i-1}))|\mathcal{F}(t_{i-1}) \sim \text{Poisson}(\lambda_{\pm}(t_{i-1})(t - t_{i-1}))$, $t_{i-1} \leq t \leq t_i$. 
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$$S(t) = S(0) \exp \{\delta(N_+(t) - N_-(t))\}$$

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- $(N_{\pm}(t) - N_{\pm}(t_{i-1})) | \mathcal{F}(t_{i-1}) \sim \text{Poisson}(\lambda_{\pm}(t_{i-1})(t - t_{i-1}))$, $t_{i-1} \leq t \leq t_i$.
- $\lambda_+(t) = \lambda_+(t_{i-1})$ and $\lambda_-(t) = \lambda_-(t_{i-1})$, $t_{i-1} \leq t < t_i$. 
Intensity model

The asset price $S(t)$ satisfies

$$S(t) = S(0) \exp \{ \delta (N_+(t) - N_-(t)) \}$$

for some constant $\delta > 0$.

- $(N_\pm(t) - N_\pm(t_{i-1}))|\mathcal{F}(t_{i-1}) \sim \text{Poisson}(\lambda_\pm(t_{i-1})(t - t_{i-1}))$, $t_{i-1} \leq t < t_i$.
- $\lambda_+(t) = \lambda_+(t_{i-1})$ and $\lambda_-(t) = \lambda_-(t_{i-1})$, $t_{i-1} \leq t < t_i$.
- $N_+(t) - N_+(t_{i-1})$ and $N_-(t) - N_-(t_{i-1})$ are conditionally independent with given $\mathcal{F}(t_{i-1})$, $t_{i-1} \leq t \leq t_i$. 
The conditional variance of log return $X(t_i)$ is given by a linear combination of intensities. More precisely,

$$\text{Var}(X(t_i)|\mathcal{F}(t_{i-1})) = \delta^2(\lambda_+(t_{i-1}) + \lambda_-(t_{i-1})).$$
Drift, correction factor and shock

Definition (Decomposition of Log-Return)
Define $\mu$ (drift), $\gamma$ (mean correction), $\varepsilon$ (shock) by

$$
\mu(t_i) = \{(e^\delta - 1)\lambda_+(t_{i-1}) + (e^{-\delta} - 1)\lambda_-(t_{i-1})\} \Delta t
$$
$$
\gamma(t_i) = \{(e^\delta - 1 - \delta)\lambda_+(t_{i-1}) + (e^{-\delta} - 1 + \delta)\lambda_-(t_{i-1})\} \Delta t
$$
$$
\varepsilon(t_i) = X(t_i) - \mathbb{E}[X(t_i)|\mathcal{F}(t_{i-1})].
$$

Then

$$
X(t_i) = \mu(t_i) - \gamma(t_i) + \varepsilon(t_i).
$$
Equivalent martingale measure

Definition (Radon–Nikodym derivative)
Take $\tilde{\lambda}_+$ and $\tilde{\lambda}_-$ such that

$$(e^\delta - 1)\tilde{\lambda}_+(t) + (e^{-\delta} - 1)\tilde{\lambda}_-(t) = r$$

and let

$$Z(T) = \exp \sum_{i=1}^{N} \left\{ \left( \lambda_+(t) + \lambda_-(t) - \tilde{\lambda}_+(t) - \tilde{\lambda}_-(t) \right) \Delta t ight.$$ 

$$+ (N_+(t_i) - N_+(t_{i-1})) \log \frac{\tilde{\lambda}_+(t_{i-1})}{\lambda_+(t_{i-1})}$$

$$+ (N_-(t_i) - N_-(t_{i-1})) \log \frac{\tilde{\lambda}_-(t_{i-1})}{\lambda_-(t_{i-1})} \right\}.$$
\[ Q(A) = \int_A Z(T) d\mathbb{P}. \]

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<thead>
<tr>
<th></th>
<th>( \mathbb{P} )</th>
<th>( \mathbb{Q} )</th>
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</thead>
<tbody>
<tr>
<td><strong>Intensities</strong></td>
<td>( \lambda_+(t_i) )</td>
<td>( \tilde{\lambda}_+(t_i) )</td>
</tr>
<tr>
<td></td>
<td>( \lambda_-(t_i) )</td>
<td>( \tilde{\lambda}_-(t_i) )</td>
</tr>
<tr>
<td><strong>Drift</strong></td>
<td>( \mu(t_i) )</td>
<td>( r \Delta t )</td>
</tr>
<tr>
<td><strong>Shock</strong></td>
<td>( \varepsilon(t_i) )</td>
<td>( \tilde{\varepsilon}(t_i) )</td>
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Remark
Asset price (return) movements model using autoregressive heteroscedasticity.

\[
h(t_i) = \text{Var}(X(t_i)|\mathcal{F}(t_{i-1})) \\
= \omega + \beta h(t_{i-1}) + \alpha \varepsilon^2(t_{i-1})
\]

\( h(t_i) \): conditional variance \\
\( \varepsilon(t_i) \): innovation \\
\( \{\omega, \beta, \alpha\} \): parameters.
Autoregressive for intensity

GARCH assumption for intensities:

\[
\begin{align*}
\lambda_+(t_i) &= \omega_+ + \beta_+ h(t_{i-1}) + \alpha_+ \varepsilon^2(t_i) \\
\lambda_-(t_i) &= \omega_- + \beta_- h(t_{i-1}) + \alpha_- \varepsilon^2(t_i)
\end{align*}
\]

implies

\[
\begin{align*}
h(t_i) &= \omega^* + \beta h(t_{i-1}) + \alpha^* \varepsilon^2(t_{i-1})
\end{align*}
\]

if \( \beta_+ = \beta_- \).
Maximum likelihood estimation

The joint distribution of $X_1, \ldots, X_n$ with a parameter set $\theta$ is given by

$$f_\theta(x_1, \ldots, x_n | \lambda_\pm(t_0)) = f_\theta(x_1 | \lambda_\pm(t_0)) f_\theta(x_2 | \lambda_\pm(t_1)) \times \cdots \times f_\theta(x_n | \lambda_\pm(t_{n-1}))$$

where

$$f_\theta(x_i | \lambda_\pm(t_{i-1})) = \exp\{-\lambda_+(t_{i-1}) - \lambda_-(t_{i-1})\} \left(\frac{\lambda_+(t_{i-1})}{\lambda_-(t_{i-1})}\right)^{x_i/2\delta} \times I_{x_i/\delta}(2\sqrt{\lambda_+(t_{i-1})\lambda_-(t_{i-1})})$$

Goal: Find $\theta$ maximizing $f_\theta(x_1, \ldots, x_n | \lambda_\pm(t_0))$. 
Intensities for estimation

GJR GARCH:

\[
\lambda_{\pm}(t_i) = \omega_{\pm} + \beta_{\pm}\lambda_{\pm}(t_{i-1}) + (\alpha_{\pm} + \gamma_{\pm}I(t_i))\varepsilon^2(t_i)
\]

\[
\lambda_{\pm}(t_i) = \omega_{\pm} + \beta_{\pm}\lambda_{\pm}(t_{i-1}) + (\alpha_{\pm} + \gamma_{\pm}I(t_i))\varepsilon^2(t_i)
\]

where

\[
I(t_i) = \begin{cases} 
1, & \varepsilon(t_i) < 0 \\
0, & \varepsilon(t_i) \geq 0. 
\end{cases}
\]
Estimates

Note that asset price and intensities are

\[ S(t) = S(0) \exp \{ \delta (N_+(t) - N_-(t)) \} , \]

\[ \lambda_+(t_i) = \omega_+ + \beta_+ \lambda_+(t_{i-1}) + (\alpha_+ + \gamma_+ I(t_i)) \varepsilon^2(t_i), \]
\[ \lambda_-(t_i) = \omega_- + \beta_- \lambda_-(t_{i-1}) + (\alpha_- + \gamma_- I(t_i)) \varepsilon^2(t_i). \]

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<tr>
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<th>( \delta = 2.0 \times 10^{-3} )</th>
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<tr>
<td>( \omega_+ )</td>
<td>8.50 \times 10^{-2}</td>
</tr>
<tr>
<td>( \beta_+ )</td>
<td>9.39 \times 10^{-1}</td>
</tr>
<tr>
<td>( \alpha_+ )</td>
<td>9.79 \times 10^{2}</td>
</tr>
<tr>
<td>( \gamma_+ )</td>
<td>1.09 \times 10^{4}</td>
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Volatility clustering (right).

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Conditional correlation(2)

On the condition of current return’s sign:

\[
\text{Corr}(X_t, |X_{t-\ell}| \mid X_t > 0) \quad \text{and} \quad \text{Corr}(X_t, |X_{t-\ell}| \mid X_t < 0)
\]
Concluding remark

- Conditional asymmetries of stock returns responding to the past information.
- Poisson intensity model as a new approach for describing asset returns.
- Linkage between GARCH and intensity model.
- Issues on measure changes and martingale methods for derivative pricing.
- Estimation results and conditional asymmetries.
Thank you!