On a Heath-Jarrow-Morton approach for stock markets

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Table of content

1. Introduction
2. The philosophy behind HJM
3. Setting Lévy in motion
4. An examples
Introduction
Consider a market with a canonical reference asset and some derivatives based on it.

If we model the canonical reference asset in detail under the martingale measure then the prices of the derivatives are given by conditional expectation.

If the derivatives are traded liquidly then the model prices may contradict the prices observed on the real market.

First way out: Calibration

2nd way out: Model the derivatives directly

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The philosophy behind HJM
Summary of the situation

- There is a canonical reference asset
- There are some derivatives based on it
- We want to model the derivatives
- It is hard to model the derivatives directly, because they have complicated dependencies on each other

- Find a reparametrisation to get rid of the complicated dependencies (codebook)
- Model the codebook-process directly, such that all the derivatives are martingales under the martingale measure. (no arbitrage condition)
- How does the canonical reference asset look like? (consistency condition)
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- Choose a class of models (simple models) such that
  - The simple models do not allow for arbitrage.
  - The class has a simple parameter space \( C \)
  - There is a (simple) one to one function \( \Phi \) which maps a given parameter and a given price of the underlying to the price of the derivative.

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Example I: Heath et. al (1992)

- Canonical reference asset: Money market account
  \[ dK(t) = K(t)r(t)\,dt \]

- Liquid derivatives: Bonds \( B(t, T) \) with \( B(T, T) = 1 \).
- Simple model: \( dK(t) = K(t)r(t)\,dt \) with deterministic short rate \( r(t) \)
  - In this setup: \( B(t, T) = \exp(-\int_t^T r(s)\,ds) \)
  - Conversely: \( r(T) = -\partial_T \log(B(t, T)) \)

This inspires the codebook \( f(t, T) = -\partial_T \log(B(t, T)) \)

- Model the dynamics \( df(t, T) = \alpha(t, T)\,dt + \beta(t, T)\,dW(t) \)
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- Canonical reference asset: Stock \( S(t) \)
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- Simple model: Purely discontinuous time-inhomogenous exponential Lévy processes
  - In this setup: \( C(t, T, K) \) can be obtained from the Lévy density \( K(t, u) \) via a function \( \Phi \).
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- Model the dynamics \( X(t, T, u) = \alpha(t, T, u)dt + \beta(t, T, u)dW(t) \)
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Setting Lévy in motion.
- Canonical reference asset: Stock $S(t) = \exp(X(t))$ with return $X$.
- Liquid derivatives: Calls $C(t, T, K)$ with maturity $T$ and strike $K$. 
Simple model: exponential time-inhomogenous Lévy processes

\[ E(e^{iuX_t}) = \exp \left( \int_0^t \psi^X(s, u) ds \right) \]

where \( \psi^X(s, u) \) is given by a generalised Lévy-Khintchine formula.

- In this setup: \( C(t, T, K) \) can be obtained by Fourier technics from \( \psi^X \) by a formula \( \Phi^{-1} \) which can be found in (Belomestny.Reiss 99)

\[ O(t, T, x) = \mathcal{F} \left\{ u \rightarrow \frac{1 - \exp \left( \int_0^t \psi^X(s, u) ds \right)}{u^2 + iu} \right\} (x), \]

\[ C(t, T, K) = (S(t) - K)^+ + K O \left( t, T, \log \left( \frac{K}{S(t)} \right) \right). \]

- Conversely: \( \psi^X(T, u) \) can be obtained from the call option prices

\[ O(t, T, x) := e^{-(x+X(t))} C \left( t, T, e^{x+X(t)} \right) - (e^{-x} - 1)^+ \]

\[ \psi^X(T, u) = \partial_T \log \left( 1 - (u^2 + iu) \mathcal{F}\{x \rightarrow O(t, T, x)\}(u) \right) \]
This inspires the codebook $\Psi(t, T, u) := \Phi^{-1}(K \mapsto C(t, T, K))(u)$

Model the dynamics $\Psi(t, T, u) = \alpha(t, T, u)dt + \beta(t, T, u)dL(t)$

Drift condition: $\alpha(t, T, u) = \partial_T \psi^L \left( \int_t^T \beta(t, r, u)dr \right)$

Consistency condition: $\psi^X(t, u) = \Psi(t, t, u)$
An example
A deterministic example

- We consider the situation: \( S(t) = \exp(X(t)) \)
- \( d\Psi(t, T, u) = \alpha(t, T, u)dt + \beta(t, T, u)dL(t) \) for an increasing Lévy process \( L \)
- \( \beta(t, T, u) = \frac{u^2 - iu}{2} e^{\lambda(T-t)} \) for some \( \lambda \in \mathbb{R}_+ \)
- The conditions imply

\[
\begin{align*}
    dX(t) &= dM(t) - v(t)dt + \sqrt{v(t)}dW(t) \\
    dv(t) &= -\lambda v(t)dt + dL(t)
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\]

for some time inhomogeneous Lévy process \( M \).
It is some kind of Barndorff-Nielsen & Shephard (2001) stochastic volatility model.
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Thank you for your attention.