Better than Dynamic Mean-Variance Policy in Market with ALL Risky Assets

Xiangyu Cui and Duan Li

Department of Systems Engineering & Engineering Management
The Chinese University of Hong Kong

June 15, 2010
Outline

1 Introduction

2 Discrete-time Dynamic Mean-Variance Portfolio Selection

3 Pseudo Efficiency and Revised Policies

4 Conclusions
Outline

1. Introduction
2. Discrete-time Dynamic Mean-Variance Portfolio Selection
3. Pseudo Efficiency and Revised Policies
4. Conclusions
Better than Dynamic Mean-Variance Policy in Market with ALL Risky Assets

Introduction

Dynamic Mean-Variance Portfolio Selection

- In the early 1950s, Markowitz published his pioneering work on single-period mean-variance portfolio selection, which has paved a foundation of modern financial analysis.
- [Li and Ng 2000] solved the mean-variance formulation of the multi-period portfolio selection problem by adopting an embedding scheme. In the same year, [Zhou and Li 2000] also solved the mean-variance formulation in continuous-time by adopting the same embedding scheme.
- [Zhu, Li and Wang 2003] investigated the wealth reduction phenomena associated with the optimal multi-period mean-variance policy. [Basak and Chabakauri 2008] also recognized that investors may have incentives to deviate from the optimal dynamic mean-variance policy, which is termed pre-committed optimal policy, before reaching the terminal time.
Time consistent dynamic risk measure


- Although “Time Consistency” requirements for dynamic risk measure introduced by [Rosazza Gianin 2002], [Boda and Filar 2006], [Artzner, Delbaen, Eber, Heath, Ku 2007], [Jobert and Rogers 2008] read differently, they all have their essence rooted in Bellman’s dynamic programming.

- [Cui, Li, Wang and Zhu 2009] introduced the concept of Time Consistency in Efficiency for mean-risk model, which is rooted in multi-objective dynamic programming, and derived a better revised mean-variance policy in markets with a riskless asset.
Time consistency in efficiency

Definition (Time Consistency in Efficiency)
Assume that \((\pi_0^*, \ldots, \pi_{T-1}^*)\) is the optimal policy of

\[
\min_{\pi_0, \ldots, \pi_{T-1}} \left\{ \mathcal{M}_{0-T}(\pi_0, \ldots, \pi_{T-1} \mid x_0) + \lambda E(x_T \mid \pi_0, \ldots, \pi_{T-1}, x_0) \right\}, \quad \lambda \leq 0.
\]

Risk measure \(\mathcal{M}\) (and its overall optimal policy) is said to satisfy time consistency in efficiency, if for all \(t = 1, \ldots, T - 1\),

\[
(\pi_t^*, \ldots, \pi_{T-1}^*) \in \arg \min_{\pi_t, \ldots, \pi_{T-1}} \left\{ \mathcal{M}_{t-T}(\pi_t, \ldots, \pi_{T-1} \mid x_t) + \lambda_t E(x_T \mid \pi_t, \ldots, \pi_{T-1}, x_t) \right\},
\]

holds for some nonpositive \(\lambda_t\) and any possible wealth level \(x_t\).
Outline

1. Introduction

2. Discrete-time Dynamic Mean-Variance Portfolio Selection

3. Pseudo Efficiency and Revised Policies

4. Conclusions
Market Setting

- Consider a capital market consisted of only $n + 1$ risky assets within a finite time horizon $T$.
- $e_t = (e_0^t, \ldots, e_n^t)'$: the vector of random total return rates of the $n + 1$ risky assets during period $t$ with known first two moments, the mean and the covariance.
- Vectors $e_t$, $t = 0, 1, \ldots, T - 1$, are assumed to be statistically independent.
- $x_0$: a given initial wealth level, .
- $x_t$: the wealth level at the beginning of the $t$-th time period.
- $u_t^i$ ($i = 1, 2, \ldots, n$): the amount invested in the $i$th risky asset at the beginning of the $t$-th time period.
Problem Formulation

The dynamic mean-variance portfolio problem is given by

\[
(MV) \quad \min \quad \text{Var}(x_T|x_0) + \lambda \text{E}(x_T|x_0)
\]
\[
\text{s.t.} \quad x_{t+1} = e_0^tx_t + P'_tu_t, \quad t = 0, 1, \ldots, T - 1, \quad (1)
\]
\[
x_0 > 0 \text{ is given,}
\]

where

\[
P_t = (P^1_t, P^2_t, \ldots, P^n_t)' = ((e^1_t - e^0_t), (e^2_t - e^0_t), \ldots, (e^n_t - e^0_t))'
\]
satisfies

\[
E(P_tP'_t) > 0, \quad \forall t = 0, 1, \ldots, T - 1,
\]
\[
E((e_t^0)^2) - E(e^0_tP'_t)E^{-1}(P_tP'_t)E(e^0_tP_t) > 0, \quad \forall t = 0, 1, \ldots, T - 1.
\]
Pre-committed Optimal Policy

The pre-committed optimal policy for \((MV)\) [Li and Ng 2000]:

\[
u_t^*(x_t) = -E^{-1}(P_t P'_t)E(e_0^0 P_t) x_t + \Gamma \left( \frac{\mu_{t+1}}{\tau_{t+1}} \right) E^{-1}(P_t P'_t) E(P_t).
\]

where \(\Gamma = \frac{1}{2} \left( b_0 x_0 - \frac{\nu_0 \lambda}{2a_0} \right)\) is termed risk attitude parameter. Furthermore, [Li and Ng 2000] give the minimum variance set of \((MV)\) explicitly as follows,

\[
Var(x_T|x_0) = \frac{a_0}{\nu_0^2} (E(x_T|x_0) - (\mu_0 + b_0 \nu_0) x_0)^2 + c_0 x_0^2. \tag{2}
\]

It is easy to verify that, when \(E(x_T|x_0) \geq (\mu_0 + b_0 \nu_0) x_0\), the mean-variance pair is efficient.
Parameters

Define the following parameters:

\[ B_t = E\left(\mathbf{p}_t'\right)E^{-1}\left(\mathbf{p}_t\mathbf{p}_t'\right)E\left(\mathbf{p}_t\right) > 0, \]
\[ A^1_t = E\left(e^0_t\right) - E\left(\mathbf{p}_t'\right)E^{-1}\left(\mathbf{p}_t\mathbf{p}_t'\right)E\left(e^0_t\mathbf{p}_t\right), \]
\[ A^2_t = E\left((e^0_t)^2\right) - E\left(e^0_t\mathbf{p}_t'\right)E^{-1}\left(\mathbf{p}_t\mathbf{p}_t'\right)E\left(e^0_t\mathbf{p}_t\right) > 0, \]
\[ \mu_t = \prod_{k=t}^{T-1} A^1_t, \quad \nu_t = \sum_{k=t}^{T-1} \left( \prod_{j=k+1}^{T-1} A^1_j \right) B^1_k, \quad \tau_t = \prod_{k=t}^{T-1} A^2_k, \]
\[ a_t = \frac{\nu_t}{2} - (\nu_t)^2, \quad b_t = \frac{\mu_t \nu_t}{a_t} = \frac{2\mu_t}{1 - 2\nu_t}, \quad c_t = \tau_t - (\mu_t)^2 - a_t(b_t)^2. \]
Outline

1. Introduction

2. Discrete-time Dynamic Mean-Variance Portfolio Selection

3. Pseudo Efficiency and Revised Policies

4. Conclusions
Three dimensional objective space

The pre-committed mean-variance efficient pair, which satisfies equation (2) for \((E(x_T|x_0), Var(x_T|x_0))\), is Pareto-optimal in the objective space of

\[
\{ \max (\text{expected terminal wealth}), \\
\min (\text{variance of the terminal wealth}) \}.
\]

In the real world, we’d better consider the efficiency in an expanded three-dimensional objective space:

\[
\{ \min (\text{initial investment level}), \\
\max (\text{expected terminal wealth}), \\
\min (\text{variance of the terminal wealth}) \}.
\]
Pseudo Efficiency (Type 1)

Consider the revised mean-variance portfolio selection,

\[
\begin{align*}
(RMV_1) \quad & \min \ Var(x_T|y_0) + \lambda E(x_T|y_0) \\
& \text{s.t.} \quad x_{t+1} = e_t^0 x_t + P_t' u_t, \quad t = 1, 2, \ldots, T - 1, \\
& \quad x_1 = e_0^0 y_0 + P_0' u_0, \\
& \quad y_0 \leq x_0.
\end{align*}
\]

Definition

For a wealth level $x_0$, if an efficient mean-variance pair for $(MV)$ is dominated by a mean-variance pair of problem $(RMV_1)$, i.e.,

\[
(-x_0, E(x_T|x_0), -Var(x_T|x_0)) \prec (-y_0, E(x_T|y_0), -Var(x_T|y_0)),
\]

the given $T$-period mean-variance pair is termed pseudo efficient (type 1).
The Existence of Pseudo Efficiency (Type 1)

Proposition

Pseudo efficiency condition (3) $\iff x_0 > \bar{x}_0^* = \Gamma \mu_0 / \tau_0$.

For a given positive initial wealth $x_0$, condition $x_0 > \bar{x}_0^*$ does not hold when

$$\lambda = \begin{cases} 
\leq \frac{2(\mu_0^2 - (1 - 2\nu_0))}{\mu_0} x_0 < 0, & \text{if } \mu_0 > 0, \\
\geq \frac{2(\mu_0^2 - (1 - 2\nu_0))}{\mu_0} x_0 > 0, & \text{if } \mu_0 < 0.
\end{cases}$$

Remark

The concept of pseudo efficiency (type 1) can be extended to truncated $(T - s)$-period problem.
The First Type of Revised Policies

Proposed $T$-period revised portfolio policy for $(MV)$:

$$
\hat{u}_k(\hat{x}_k) = -E^{-1}(P_kP'_k)E(e_0^0P_k)\hat{x}_k + \Gamma_k \left( \frac{\mu_{k+1}}{\tau_{k+1}} \right) E^{-1}(P_kP'_k)E(P_k); \quad (4)
$$

$$
\hat{x}_k = \begin{cases} 
\bar{x}_k, & \text{if } \bar{x}_k \leq \bar{x}_k^*, \\
-\bar{x}_k + \frac{2\mu_k(\mu_k\bar{x}_k + 2\nu_k\Gamma_{k-1})}{2\nu_k\tau_k + \mu_k^2}, & \text{if } \bar{x}_k > \bar{x}_k^*,
\end{cases}
$$

$$
\bar{x}_0 = x_0
$$

$$
\bar{x}_{k+1} = e_0^0\hat{x}_k + P'_k\hat{u}_k(\hat{x}_k),
$$

$$
\Gamma_k = \begin{cases} 
\Gamma_{k-1}, & \text{if } \bar{x}_k \leq \bar{x}_k^*, \\
\Gamma_{k-1} + \frac{2\mu_k\tau_k(\bar{x}_k - \bar{x}_k^*)}{2\nu_k\tau_k + \mu_k^2}, & \text{if } \bar{x}_k > \bar{x}_k^*,
\end{cases}
$$

$$
\Gamma_{-1} = \frac{1}{2} \left( b_0x_0 - \frac{\lambda_0\nu_0}{2a_0} \right)
$$

$$
\bar{x}_k^* = \frac{\Gamma_{k-1}\mu_k}{\tau_k}.
$$
Scheme Illustration of the First Revised Policy

Figure: The scheme of the first revised policy

(a) \( \mu_k > 0 \)

(b) \( \mu_k < 0 \)
Performance

- The first type of revised policies keeps the conditional mean and variance unchanged, thus achieving the same mean-variance pair as does the pre-committed optimal mean-variance policy of the $T$-period problem ($MV$), while having a possibility to take positive free cash flow stream, $\{\bar{x}_k - \hat{x}_k\}$, out of the market during the investment process, i.e.,

\[
E(\bar{x}_T|x_0)|\hat{u}^* = E(x_T|x_0)|u^*,
\]
\[
Var(\bar{x}_T|x_0)|\hat{u}^* = Var(x_T|x_0)|u^*,
\]
\[
P\{\bigcup_{k=1}^{N-1} [(\bar{x}_k - \hat{x}_k)^+ > 0] \mid x_0 \} > 0.
\]
Pseudo Efficiency (Type 2)

Consider the revised mean-variance portfolio selection,

\[(RMV_2) \quad \min \ Var(x_T + x_0 - y_0|x_0) + E(x_T + x_0 - y_0|x_0)\]
\[\text{s.t.} \quad x_{t+1} = e^0_t x_t + P'_t u_t, \quad t = 1, 2, \ldots, T - 1,\]
\[x_1 = e^0_0 y_0 + P'_0 u_0,\]
\[y_0 \leq x_0.\]

Definition

For a wealth level \(x_0\), if an efficient mean-variance pair for (MV) is not pseudo efficient (type 1) and is, however, dominated by a total mean-variance pair of problem (RMV_2), i.e.,

\[(E(x_T|x_0), -Var(x_T|x_0)) \prec (E(x_T + x_0 - y_0|x_0), -Var(x_T + x_0 - y_0|x_0)),\]

then the given \(T\)-period mean-variance pair is called pseudo efficient (type 2).
The Existence of Pseudo Efficiency (Type 2)

Proposition

Pseudo efficiency (type 2) condition

\[ (\tau_0 - \mu_0)x_0 > (\mu_0 - 1 + 2\nu_0)\Gamma. \]

For a given positive initial wealth \( x_0 \), condition

\( (\tau_0 - \mu_0)x_0 > (\mu_0 - 1 + 2\nu_0)\Gamma \) does not hold when

\[ \lambda = \begin{cases} 
\leq \frac{2(\mu_0^2 - \tau_0(1 - 2\nu_0))}{\mu_0 - 1 + 2\nu_0} x_0 < 0, & \text{if } \mu_0 > 1 - 2\nu_0, \\
\geq \frac{2(\mu_0^2 - \tau_0(1 - 2\nu_0))}{\mu_0 - 1 + 2\nu_0} x_0 > 0, & \text{if } \mu_0 < 1 - 2\nu_0.
\end{cases} \]
The Second Type of Revised Policies

Proposed $T$-period revised portfolio policy, $\tilde{u}_k^*(\tilde{x}_k)$, $k = 0, \ldots, T - 1$:

$$\tilde{u}_k^*(\tilde{x}_k) = -E^{-1}(P_kP'_k)E(e_0^0P_k)\tilde{x}_k + \Gamma_k \left( \frac{\mu_k + 1}{\tau_k + 1} \right) E^{-1}(P_kP'_k)E(P_k); \quad (5)$$

$$\tilde{x}_k = \begin{cases} \tilde{x}_k, & \text{if } (\tau_k - \mu_k)\tilde{x}_k \leq (\mu_k - 1 + 2\nu_k)\Gamma_{k-1}, \\ \frac{(\mu_k - 1 + 2\nu_k)[(\mu_k - 1)\tilde{x}_k + 2\nu_k\Gamma_{k-1}]}{2\nu_k(\tau_k - 1) + (\mu_k - 1)^2}, & \text{if } (\tau_k - \mu_k)\tilde{x}_k > (\mu_k - 1 + 2\nu_k)\Gamma_{k-1}, \end{cases}$$

$$\tilde{x}_0 = x_0$$

$$\tilde{x}_{k+1} = e_0^0\tilde{x}_k + P'_k\tilde{u}_k^*(\tilde{x}_k),$$

$$\Gamma_k = \begin{cases} \Gamma_{k-1}, & \text{if } (\tau_k - \mu_k)\tilde{x}_k \leq (\mu_k - 1 + 2\nu_k)\Gamma_{k-1}, \\ \frac{(\tau_k - \mu_k)[(\mu_k - 1)\tilde{x}_k + 2\nu_k\Gamma_{k-1}]}{2\nu_k(\tau_k - 1) + (\mu_k - 1)^2}, & \text{if } (\tau_k - \mu_k)\tilde{x}_k > (\mu_k - 1 + 2\nu_k)\Gamma_{k-1}, \end{cases}$$

$$\Gamma_{-1} = \frac{1}{2} \left( b_0x_0 - \frac{\lambda_0\nu_0}{2a_0} \right).$$
Scheme Illustration of the Second Revised Policy

(a) $\mu_k > 0$

(b) $\mu_k < 0$

Figure: The scheme of the second revised policy
Performance

- Denote $\Delta \tilde{x}_k = \tilde{x}_k - \check{x}_k$.
- The second type of revised policies achieves the same total mean as the pre-committed optimal mean-variance policy of the $T$-period problem ($MV$) does, while having smaller total variance than the pre-committed optimal policy does, i.e.,

$$E(\tilde{x}_T + \sum_{j=0}^{T-1} \Delta \tilde{x}_j | x_0) | \bar{u}^* = E(x_T | x_0) | u^*,$$

$$Var(\tilde{x}_T + \sum_{j=0}^{T-1} \Delta \tilde{x}_j | x_0) | \bar{u}^* < Var(x_T | x_0) | u^*.$$
Outline

1. Introduction
2. Discrete-time Dynamic Mean-Variance Portfolio Selection
3. Pseudo Efficiency and Revised Policies
4. Conclusions
Conclusion

• The dynamic mean-variance portfolio selection in markets with all risky assets is not time consistent in efficiency, due to the inherent nonseparable nature of the involved variance term.

• By adding the initial investment level into the objective space, the concept of pseudo efficiency (type 1 or type 2) has been introduced.

• By relaxing the self-financing constraint, two revised policies have been proposed to tackle pseudo efficiency (type 1 or type 2), thus achieving better performance than the original dynamic mean-variance policy.
Thank you for your attention!